Math 511

$\boxed{Q's}$ p. 293 Eigen Value / Vector / Space

example #5 $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

① $A - \lambda I \to \begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix}$

② λ_i? eigen values

$$\det\begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = 0 \to (1-\lambda)^2 + 4 = 0$$

$$\to (1-\lambda)^2 = -4 \to (1-\lambda) = \pm 2i$$

$$\lambda = 1 \pm 2i$$

$$\lambda_1 = 1+2i \qquad \lambda_2 = 1-2i$$

③ for each λ_i find X_i

a) $\lambda_1 = 1+2i$

$$\left[\begin{array}{cc|c} 1-(1+2i) & 2 & 0 \\ -2 & 1-(1+2i) & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] i$$

$r_1 \cdot i = \text{New } r_1$

$$\to \left[\begin{array}{cc|c} 2 & 2i & 0 \\ -2 & -2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \to X_1 = -\alpha i$$

free $\boxed{X_2 = \alpha}$

$$X = \begin{bmatrix} -\alpha i \\ \alpha \end{bmatrix}$$

$$X = \alpha \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$\boxed{\lambda_1 = 1+2i \qquad X_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}}$$

if $\alpha = i$ $\quad X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1+0i \\ 0+i \end{bmatrix}$

$$\boxed{\lambda_2 = 1-2i \quad x_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}}$$

book $x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

ex $\boxed{\lambda_1 = 1+2i}$ $x_1 = \boxed{\begin{bmatrix} 1-i \\ 2i \end{bmatrix}}$

or $x_1 = (1+i)\begin{bmatrix} 1-i \\ 2i \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -2+2i \end{bmatrix}}$

pick $\boxed{\begin{bmatrix} 1 \\ -1+i \end{bmatrix}}$

ex
$$\begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \\ 2-\lambda & -3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -2-\lambda & 1 \\ 0 & -1+\lambda & 1-\lambda \\ 2-\lambda & -3 & 1 \end{vmatrix} = 1\begin{vmatrix} -1+\lambda & 1-\lambda \\ -3 & 1 \end{vmatrix} + (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -1+\lambda & 1-\lambda \end{vmatrix}$$

$$= \left[(-1+\lambda)+3(1-\lambda)\right] + (2-\lambda)\left[(-2-\lambda)(1-\lambda) - (-1+\lambda)\right.$$

$$= \cdots = -\lambda(\lambda-1)^2$$

$\boxed{6.2}$ Systems of Linear Diff. eqn's

(1) 1st order $Y' = AY \rightarrow$

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ \vdots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}}_{Y'} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}}_{\boxed{A}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{Y}$$

if A has $\{\lambda_1, x_1\}, \{\lambda_2, x_2\}, \ldots, \{\lambda_n, x_n\}$ eigen value/vect.

$\boxed{\text{Soln}}$ $Y = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + \cdots + c_n e^{\lambda_n t} x_n$

Initial Value $Y(0) = Y_0$ (@ $t=0$ told $y_1(0) = \Delta$, $y_2(0) \ldots$) initial value

$\rightarrow$ use these to solve for $c_1, c_2, \ldots, c_n$

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(2) Higher order? (ex) $y_1'' = 2y_1 + 3y_2 + y_1' - 2y_2'$

(ex) $y_1''' = y_1 - y_2 + 0y_1' + 2y_2' + 3y_1'' + 0y_2''$

$y_2''' = 2y_1 + 0y_2 + y_1' + 0y_2' + y_1'' + y_2''$

$$\boxed{Y'''} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{Y} + \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{Y'} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix}}_{Y''}$$

any thing like: $Y^{(n)} = A_1 Y + A_2 Y' + A_3 Y'' + \dots + A_m Y^{(m-1)}$

can be turned into a 1^{st} order diff eqn's

$\rightarrow \; y_1, y_2, \dots, y_m, \; y_1' = y_{m+1}, \; y_2' = y_{m+2}, \; \dots$

Partitioned Matrix $\quad Y_1 = Y, \; Y_2 = Y', \; Y_3 = Y^{(2)} = Y_2', \; \dots, \; Y_m = Y_{m-1}'$

$$\begin{bmatrix} Y_1' \\ Y_2' \\ \vdots \\ Y_m' \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & 0 \cdots & 0 \\ \vdots & & & & \\ 0 & - & - & 0 & I \\ A_1 & A_2 & A_3 & \cdots & A_m \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix}$$

ex

$$\boxed{Y'''} = \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}}_{A_2} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \underbrace{\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}}_{A_3} \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix}$$

$$\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A_1 & A_2 & A_3 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & -1 & 0 & 2 & 3 & 0 \\ 2 & 0 & -1 & 0 & 1 & 1 \end{array}\right]$$

eigen value / vectors?

A has eigen values λ_i, x_i

$$\underbrace{A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}}_{X} = \begin{bmatrix} A x_1 & A x_2 & \cdots & A x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{D}$$

So $AX = XD$

$$\boxed{A = X D X^{-1}}$$