Math 511

Q's

Review:

From Exam 1 don't worry about 7, 9, 11

From Exam 2 don't worry about 1, 2, 6

From Exam 3 don't worry about 2, 5, 10

$A \xrightarrow{\text{find}}$ eigen values, eigen vectors λ_1, x_1; λ_2, x_2; $\vdots$; λ_n, x_n

let $X = [x_1, x_2, \ldots x_n]$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

then: $AX = XD$

if X^{-1} exists, then

a) $A = XDX^{-1}$ (factorization)

or

b) $D = X^{-1}AX$ (diagonalized A)

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots \ \lambda_n \end{bmatrix}$$

- A, D are similar
- represent the same linear operator

Note: thm if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct $\rightarrow$ x_i are lin. ind.

what about if they are not distinct?

$\rightarrow$ check if x_i are lin. ind. by -det?

- non-trivial sols to lin. combo.

etc

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{-1} & 0 & 0 \\ 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{1/2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = A$$

ex

Markov Process: $V = \begin{bmatrix} \% \\ \% \\ \vdots \end{bmatrix}$ ← % of population

sum values V = 1

- V is a probability vector.

- A, markov process $V_k = A V_{k-1}$

or

$V_k = A^k V_0$

if $A = X D X^{-1}$

$$A^k = (X D \underbrace{X^{-1})(X}_{I} D \underbrace{X^{-1})}_{I} \cdots \underbrace{(X}_{I} D X^{-1})$$

$$A^k = X \underbrace{D D \cdots D}_{k\text{-times}} X^{-1} = X D^k X^{-1}$$

$$A^k = X \begin{bmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \ddots \lambda_n^k \end{bmatrix} X^{-1}$$

So

$$V_k = A^k V_0 = (X D^k X^{-1}) V_0$$

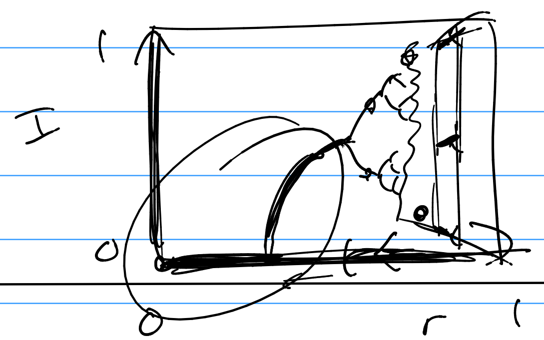
$$V_k = X \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} (X^{-1} V_0)$$

thm: $A_{n\times n}$.

if a Markov chain $\left(v_k = A^k v_0 \right)$ conv to a steady state vector x

① x is a prob. vector

② $\lambda_1 = 1$ is an eigenvalue of A and x is an eigenvector of $\lambda_1 = 1$.



$$\exp(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$\boxed{\exp(A) = e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots}$$

try this on $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$\exp(D) = \boxed{e^D = I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots}$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} + \begin{bmatrix} \frac{1}{2!}\lambda_1^2 & & & 0 \\ & \frac{1}{2!}\lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \frac{1}{2!}\lambda_n^2 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\dots & & & 0 \\ & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\dots & & \\ & & \ddots & \\ 0 & & & 1+\lambda_n+\frac{1}{2!}\lambda_n^2+\dots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{bmatrix}$$

Nice! $e^D = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \\ & & & e^{\lambda_n} \end{bmatrix}$!

$$e^A = \overset{XD^0X^{-1}}{I} + \overset{XDX^{-1}}{A} + \frac{1}{2!}\overset{XD^2X^{-1}}{A^2} + \frac{1}{3!}A^3 + \cdots$$

if $A = XDX^{-1}$

$$A^k = X D^k X^{-1}$$

$$X\underbrace{\left[I + D + \frac{1}{2!}D^2 + \cdots\right]}_{e^D}X^{-1}$$

So if $A = XDX^{-1}$

$$\boxed{e^A = Xe^DX^{-1} = X\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n}\end{bmatrix}X^{-1}}$$

So exp(A) can be found "easily" if $A = XDX^{-1}$

Why? $Y' = AY \quad Y(0) = Y_0$

guess a soln $Y = e^{tA}Y_0$

check: $Y(0) = e^{0A}Y_0 = IY_0 = Y_0$
$\quad t=0$

$$Y' = \frac{d}{dt}\left[e^{tA}Y_0\right] = Y_0\frac{d}{dt}\left[e^{tA}\right]$$

$$Y' = Y_0 \frac{d}{dt}\left[I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots \right]$$

$$= Y_0 \left[A + tA^2 + \frac{1}{2!}t^2A^3 + \dots \right]$$

$$= AY_0 \left[I + tA + \frac{1}{2!}t^2A^2 + \dots \right]$$

$$\cancel{=} AY_0 e^{tA}$$

$$= AY$$

So $Y' = AY$ ✓

$\geq$ Solve $Y' = AY \quad Y(0) = Y_0$

→ answer: $\boxed{Y = e^{tA} Y_0}$

→ $A \to \lambda_i, x_i \to X = ?$
$D = ?$

$$Y = X e^{tD} X^{-1} Y_0$$

Soln $\boxed{Y = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} X^{-1} Y_0}$