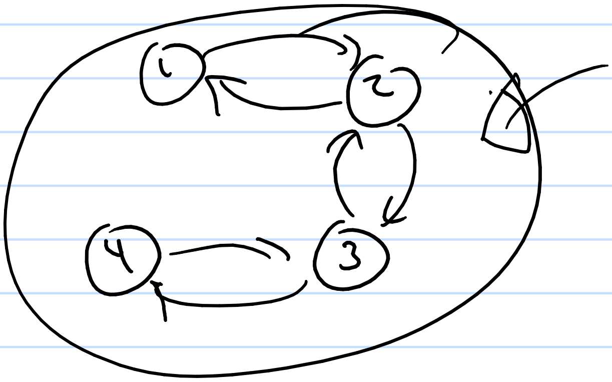
Math 322

6.4(7) on $\{1,2,3,4\}$ $\quad p = \{(a,b) \mid |a-b| = 1\}$

$p = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}$



$$M_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

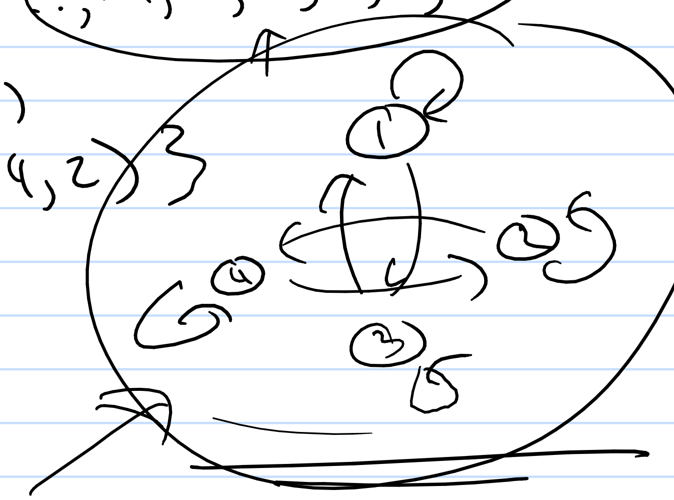
2 | #

$q = \{(a,b) \mid a-b \text{ is even}\}$

$\ldots, -4, -2, 0, 2, 4, \ldots$

$q = \{(1,1), (2,2), (3,3), (4,4),$
$(1,3), (3,1), (2,4), (4,2)\}$

$$M_q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



$\underline{pq}$, $pp = \underline{p^2}$, $qq = \underline{q^2}$

$$M_{pq} = M_p \cdot M_q$$

Properties

reflexive $\forall e\,(e\, r\, e)$

irreflexive $\forall e\,(e \not r e)$

symmetric $\forall e_1 \forall e_2\,(e_1 r e_2 \to e_2 r e_1)$

antisymmetric

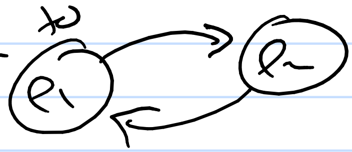
$$\forall e_1 \forall e_2\,(e_1 r e_2 \land e_2 r e_1 \to e_1 = e_2)$$

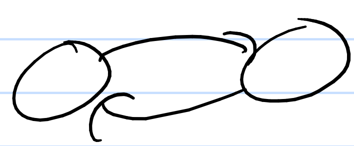
$$\equiv \forall e_1 \forall e_2\,(e_1 \neq e_2 \to \neg(e_1 r e_2 \land e_2 r e_1))$$

$$\forall \forall\ \neg(e_1 \neq e_2 \cancel{\land}\ e_1 r e_2 \land e_2 r e_1)$$

$$\equiv \neg \exists e_1 \exists e_2\,(\underline{e_1 \neq e_2} \land \underline{e_1 r e_2} \land \underline{e_2 r e_1})$$

counter to example

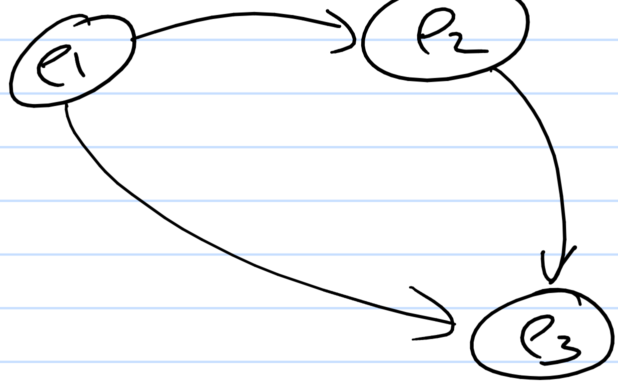




asymmetric $\forall e_1 \forall e_2\,(e_1 r e_2 \to e_2 \not r e_1)$

<u>same as</u> irreflexive <u>and</u> antisymmetric

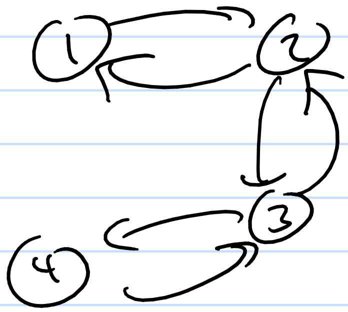
transitive



$\forall e_1 \forall e_2 \forall e_3 (e_1 r e_2 \wedge e_2 r e_3 \rightarrow e_1 r e_3)$

ex

P:



$M_P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

reflexive? No

irreflexive? Yes

sym? Yes

antisym? No

asym? No

trans? No

b/c $1r2 \wedge 2r3$

but $1 \not r 3$ $\quad 1r3 \equiv F$

Note: if you see a symmetric form and you do not see loops.

trans: $arb \wedge bra$ then ara but b/c no loops $a \not r a$

So not trans. (counterexample)

Applications:

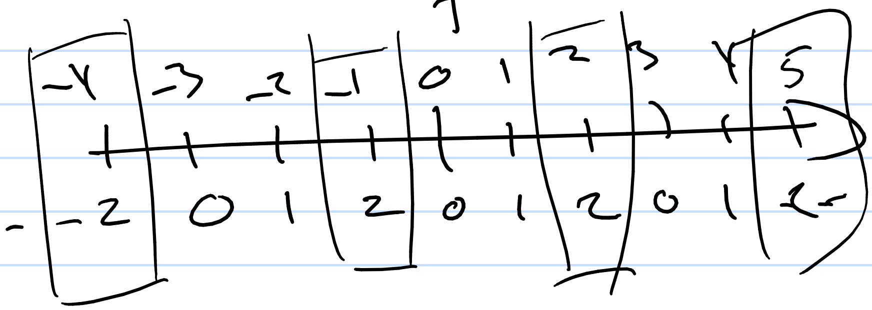
(1) Equivalence Relations.

if you show a relation is

a) reflexive
b) symmetric
c) transitive

then you call it an equivalence relation

(ex) mod 3



mod 3

Each equiv. relation will partition your set into equiv. classes

mod 3

ex

remainder 0 class $\{ \ldots, -6, -3, 0, 3, 6, 9, \ldots \}$

remainder 1 class $\{ \ldots, -5, -2, 1, 4, 7, 10, \ldots \}$

remainder 2 class $\{ \ldots, -4, -1, 2, 5, 8, 11, \ldots \}$

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ex rational number $\frac{p}{q}$ p, q are integers $q \neq 0$

(q, p have no common factors)

$$\boxed{\begin{array}{c} (p,q) \; r \; (s,t) \\ \text{iff} \quad pt = qs \end{array}}$$

$$\frac{p}{q} = \frac{s}{t}$$

check ref?

sym?

trans?

Partial Ordering

what we want is ..

given e_1, e_2, e_3 can I order them?

$e_i \leq e_j$

Need

1. reflexive
2. anti-symmetry
3. transitive

ex on integers $\leq = \{(a,b) \mid a \leq b\}$

ref?. $\forall e_1 \ (e_1 \leq e_1)$ true

antisym?. $\forall e_1 \forall e_2 (e_1 r e_2 \wedge e_2 r e_1 \rightarrow e_1 = e_2)$

$e_1 \leq e_2 \wedge e_2 \leq e_1 \rightarrow e_1 = e_2$ true

trans?. $e_1 \leq e_2 \wedge e_2 \leq e_3 \rightarrow e_1 \leq e_3$ true

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ex $d = \{(a,b) \mid a \mid b\}$

$(2,6) \in d$

$(2,5) \notin d$

$b = q \cdot a + r$ is zero?

$\frac{b}{a} = q$

ref?. true b/c $a \mid a$ is $a \cdot 1 = a$

antisym?. true $a \mid b \wedge b \mid a \rightarrow a = b$

$a \cdot q_1 = b \wedge b \cdot q_2 = a$

$a \cdot q_1 \cdot q_2 = a$ so $a \cdot 1 = b$

$q_1 \cdot q_2 = 1$ $a = b$

trans?. true

$2 < 4$
$2 < 6$
$2 < 8$
etc

but what about 4 vs 6

b/c $4 \nmid 6$ and $6 \nmid 4$

they can not be ordered versus each other.

this is why we call relations that are

ref., antisym, transitive $\boxed{\text{partial orders}}$

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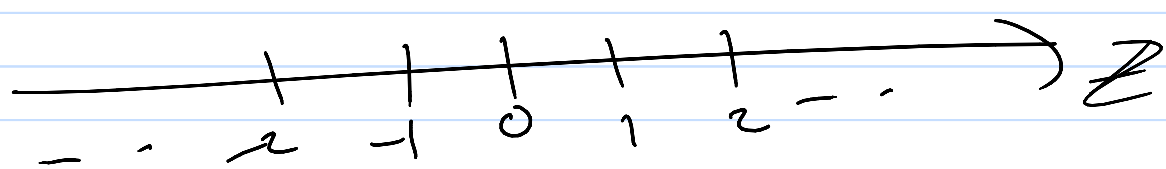
if we add the restriction to everyone must compare to everyone

→ $\boxed{\text{total order}}$

$\leq$

ex



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not good enough?

add that every subset must have a least (1^{st}) element

→ $\boxed{\text{well ordered}}$