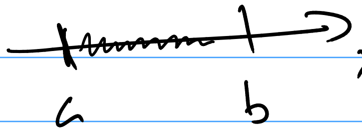
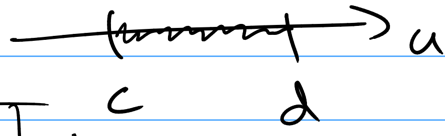
Math 3114

Change of variable / Jacobian

1D

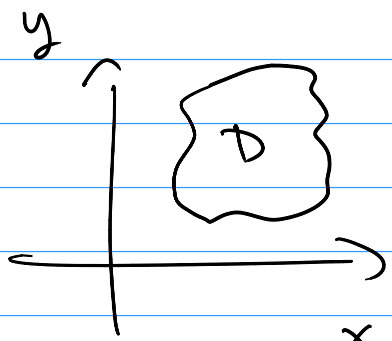


T^{-1}: $u = u(x)$



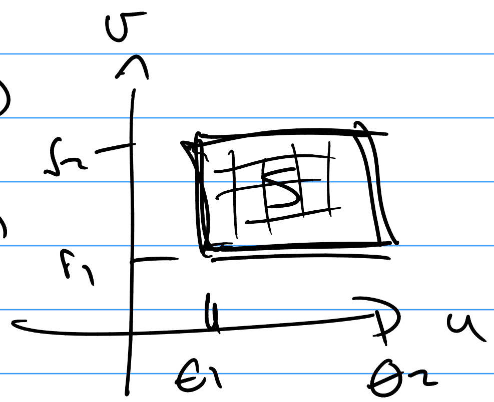
$\boxed{x = x(u)}$

2D

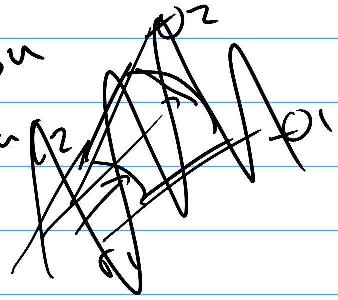


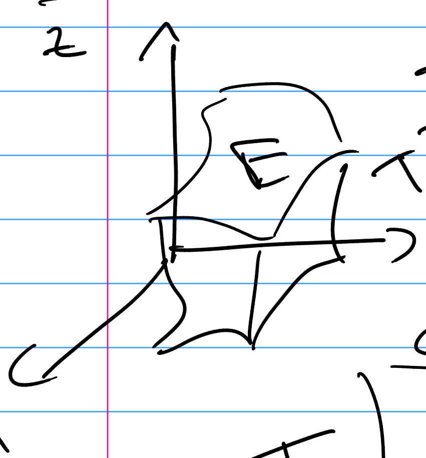
T

T^{-1} $\left| \begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array} \right.$

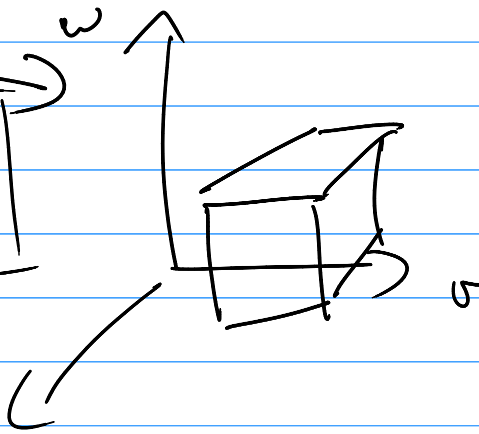


T $\left| \begin{array}{l} x = x(u,v) = v\cos u \\ y = y(u,v) = v\sin u \end{array} \right.$



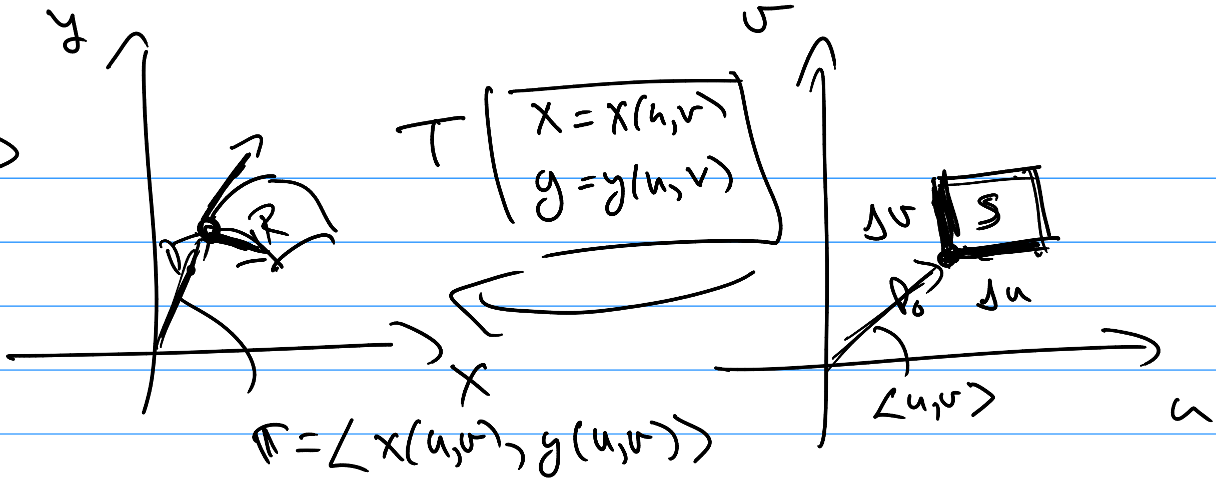


T^{-1} $\left| \begin{array}{l} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{array} \right.$

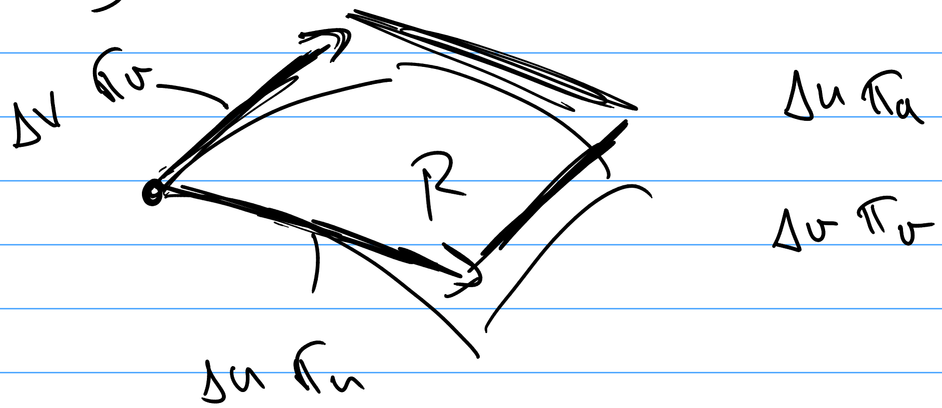


T $\left| \begin{array}{l} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{array} \right.$

$\boxed{2D}$



Scheme: $\iint_D f(x,y)\,dA = \iint ???$ u's ??? v's



area $R \approx |\Delta v\, \mathbf{r}_v \times \Delta u\, \mathbf{r}_u|$

as $\Delta v, \Delta u \to 0$ area $= |\Delta v\, \mathbf{r}_v \times \Delta u\, \mathbf{r}_u|$

$dA =$

$$dA = |\Delta v\, \mathbf{r}_v \times \Delta u\, \mathbf{r}_u| = |\mathbf{r}_v \times \mathbf{r}_u|\, \Delta u\, \Delta v$$

$$= |\mathbf{r}_v \times \mathbf{r}_u|\, du\, dv$$

For cross-product 3D?

but 2D?

$$\left|\vec{r}_u \times \vec{r}_v\right| = \left\| \begin{vmatrix} i & j & k \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} \right\| = \left\langle 0, 0, \underline{\underline{x_u y_v - x_v y_u}} \right\rangle$$

$$\left\| \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\| = \left| x_u y_v - x_v y_u \right|$$

so $T \quad \begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array} \sim \left\| \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\| = \left| x_u y_v - x_v y_u \right|$

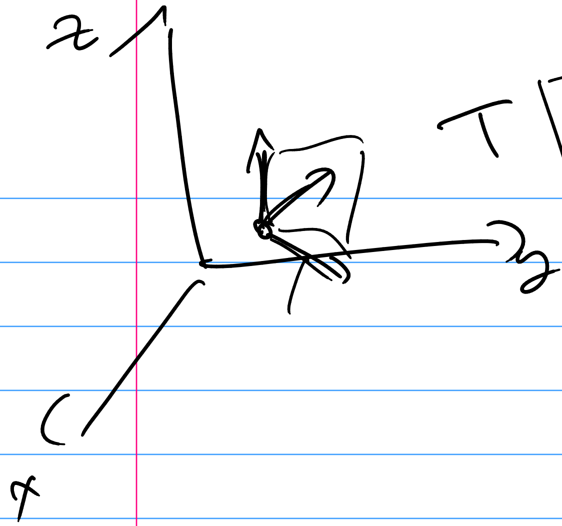
$$\underline{\underline{\text{and}}} \iint_D f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left\| \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\| du\, dv$$

Jacobian

$$T \; \boxed{\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}}$$

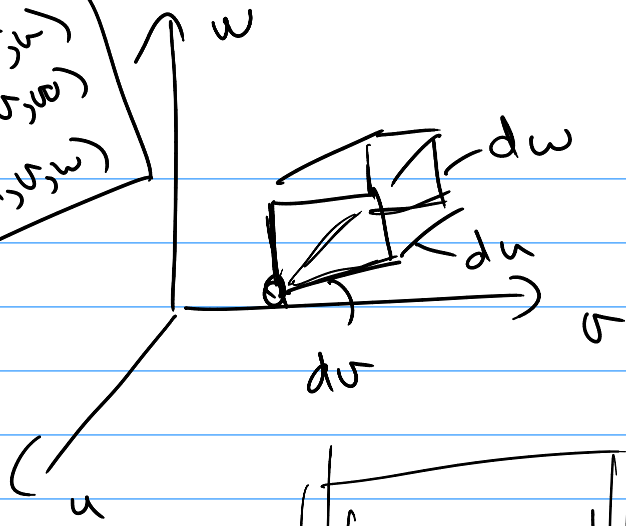
<u>Notation of Jacobian</u>

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left\| \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\|$$



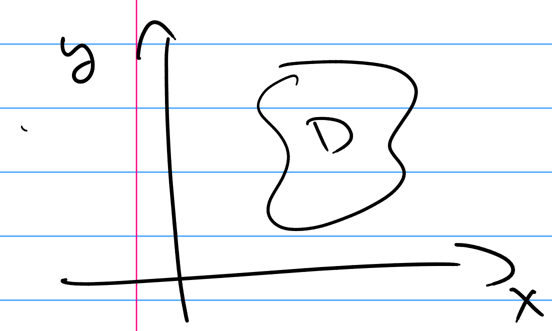
T

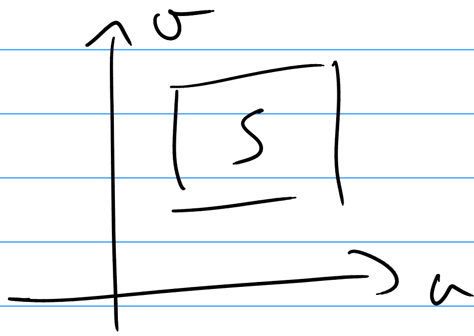
$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$



$$\iiint_E f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \right| du\,dv\,dw$$

Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$





$$x = v\cos u$$
$$y = v\sin u$$

$$\iint_D f(x,y)\,dA = \iint_S f(v\cos u, v\sin u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$x = v\cos u \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$
$$y = v\sin u$$

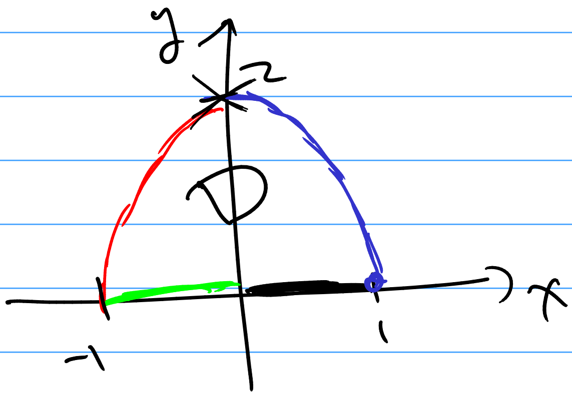
$$= \begin{vmatrix} -v\sin u & \cos u \\ v\cos u & \sin u \end{vmatrix} = -v\sin^2 u - v\cos^2 u$$
$$= -v$$

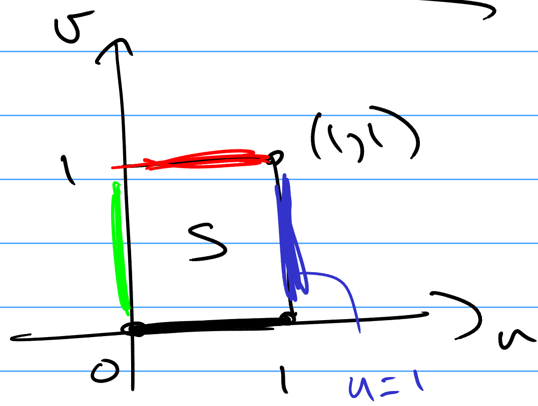
$$|-v| = v$$

$$\iint_D f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\, v\, du\, dv$$

<u>concept</u>

figure ? p. 1094





$$T \qquad x = u^2 - v^2$$
$$y = 2uv$$

$$x = 1 - v^2$$
$$y = 2v$$
$$x = 1 - \left(\frac{y}{2}\right)^2$$

$$\iint_D f(x,y)\,dA = \iint_S f(u^2-v^2, 2uv)\boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\,du\,dv$$

$$\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$