Math 344

$\boxed{\text{Final Exam}}$ 3 probs / exam (1 to 4)

so .. 12 problems in 2hrs

- Monday December 7^{th}
- WSU Schedule .. 7am to 9am

our schedule $\boxed{\text{7am to 11am}}$ (exam is open)

2 hrs to finish

- Use exam solutions to study!

(I'm taking 3 probs / exam)

Note: Exam 4 ~~10pts~~ $\boxed{\text{100pts}}$ and Syllabus

1. Required Texts
   Calculus, (eighth edition) by James Stewart
2. Exam Review on 11/20 so ...???

16.7 #3 "like example 6 p. 1172"
16.8 #1 "see probs 7,9 p. 1179"
16.9 #1 "examples 1,2 p. 1183"

0) Exam Start Time: Key

0) MyWSUid and Name:

$\mathbf{r}'(t) = \langle \cos(t), -\sin(t) \rangle$

1) Evaluate the Line Integral $\int_C \boxed{(1 + x^2 y)} ds$, where C is given by $\boldsymbol{r}(t) = \langle \sin(t), \cos(t) \rangle$ for $t = 0$ to $t = \pi$.

$x = \sin(t)$, $y = \cos(t)$

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t)) \boxed{|\mathbf{r}'(t)|\,dt}$$

$$= \int_0^{\pi} \left(1 + \sin^2(t)\cos(t)\right) \sqrt{\underbrace{\cos^2(t) + \sin^2(t)}_{=1}}\,dt$$

$$= \int_0^{\pi} \left(1 + \sin^2(t)\cos(t)\right) dt$$

$$= \boxed{\pi}$$ just use symbolic Math software

by hand

$u = \sin t \quad du = \cos t\,dt$

$$= \int_0^{\pi} 1\,dt + \int_0^{\pi} \boxed{\sin^2(t)\cos(t)}\,dt = \pi + \int_0^0 u^2\,du$$

$$= \pi + 0 = \boxed{\pi}$$

2) Evaluate the Line Integral of a Vector Field $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x, y, z) = \langle xy, yz, zx \rangle$ and C is the space curve $r(t) = \langle t, t^2, t^3 \rangle$ from $t = 0$ to $t = 1$. and $\mathbf{r}' = \langle 1, 2t, 3t^2 \rangle$

$x \quad y \quad z$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_0^1 \langle t \cdot t^2, t^2 \cdot t^3, t \cdot t^3 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$$

$$= \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt = \int_0^1 (t^3 + 5t^6)\, dt$$

$$= \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

$\langle P, Q \rangle$

3) Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $F(x,y) = \langle x^2y^3, x^3y^2 \rangle$ and C is $r(t) = \langle t^3 - 2t, t^3 + 2t \rangle$ from $t = 0$ to $t = 1$ by using the Fundamental Theorem for Line Integrals.

$r(0) = \langle 0, 0 \rangle \quad r(1) = \langle -1, 3 \rangle$

(a) $\left.\begin{array}{l} Q_x = 3x^2y^2 \\ P_y = 3x^2y^2 \end{array}\right]$ so $\langle P, Q \rangle = \langle f_x, f_y \rangle$

(b) $f_x = x^2y^3 \Rightarrow f = \frac{1}{3}x^3y^3 + C(y)$

$\frac{\partial}{\partial y}$: $f_y = x^3y^2 + C'(y)$, so $C'(y) = 0$

potential function

(c) $\boxed{f = \frac{1}{3}x^3y^3}$

$r(1) = \langle -1, 3 \rangle$: end point

$r(0) = \langle 0, 0 \rangle$: start point

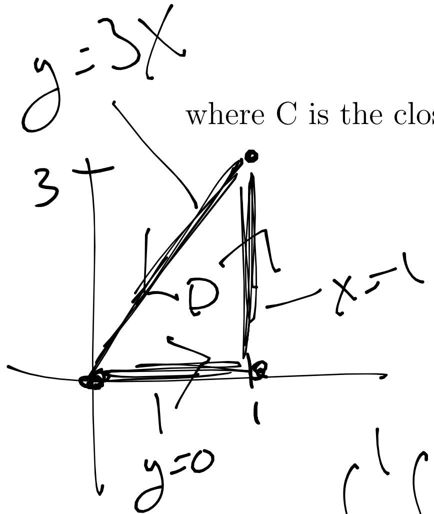
So $\int_C \nabla f \cdot d\boldsymbol{r} = \frac{1}{3}(-1)^3(3)^3 - \frac{1}{3}(0)^3(0)^3$

$= \boxed{-9}$

4) Use Green's Theorem to evaluate the Line Integral (you do not have to integrate it by hand) ...

$$\int_C \langle \underbrace{x^2y^3}_{P}, \underbrace{\sqrt{y+x^2}}_{Q} \rangle \cdot d\mathbf{r} = \iint_D (Q_x - P_y)\,dA$$

where C is the closed curve from the points (0,0) to (1,0) to (1,3) and then back to (0,0).



Q_x

$$\iint_D \left(\frac{x}{\sqrt{y+x^2}} - 3x^2y^2\right) dA$$

+10 pts

$$= \int_0^1 \int_0^{3x} \left(\frac{x}{\sqrt{y+x^2}} - 3x^2y^2\right) dy\,dx$$

???

$$= \boxed{-\frac{22}{3} + \frac{27}{4}\operatorname{arcsinh}\left(\frac{1}{\sqrt{3}}\right)} = \#$$

5) Use Green's Theorem to setup three different Line Integrals that would evaluate the Double Integral ...

$$\frac{1}{2}\left(\oint x\,dy - y\,dx\right) = \iint_D (1)\,dA = \oint x\,dy = -\oint y\,dx$$

where D is the ellipse $x^2/a^2 + y^2/b^2 = 1$ with parametric equations $x = a\cos(t)$ and $y = b\sin(t)$ with $t = 0$ to $t = 2\pi$. DO NOT SOLVE the Line Integrals.

$dx = -a\sin(t)\,dt = x'\,dt$

$dy = b\cos(t)\,dt = y'\,dt$

$$\oint x\,dy = \int_0^{2\pi} ab\cos^2(t)\,dt \quad (4)$$

$$-\oint y\,dx = \int_0^{2\pi} ab\sin^2(t)\,dt \quad (4)$$

$$\frac{1}{2}\left(\oint x\,dy - y\,dx\right) = \frac{1}{2}\int_0^{2\pi} ab\,dt \quad (4) \quad = \boxed{ab\pi}$$

not required

6) Calculate the curl of $\boldsymbol{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \boxed{\langle -2y - xy, x, yz \rangle}$$

7) Calculate the divergence $\boldsymbol{F}(x,y,z) = \langle xz - \tan(y), xyz + \cos(x), -y^2 + \tanh(xy) \rangle$

$$\nabla \cdot F = \frac{\partial}{\partial x}(xz - \tan y) = z$$

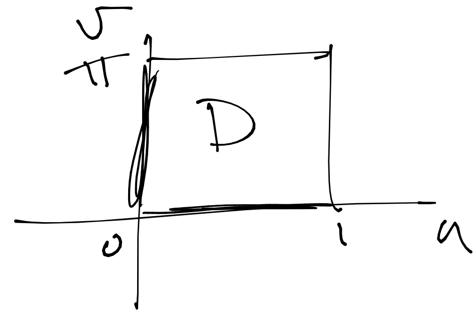
$$+ \frac{\partial}{\partial y}(xyz + \cos x) \quad + xz$$

$$+ \frac{\partial}{\partial z}(-y^2 + \tanh(xy)) \quad + 0$$

$$= \boxed{z + xz}$$

8) Setup the Double Integral to evaluate the Surface Integral $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$ for the vector field $\boldsymbol{F}(x, y, z) = \langle z, y, x\rangle$ and S is the helicoid $r(u, v) = \langle \underbrace{u\cos(v)}_{x}, \underbrace{u\sin(v)}_{y}, \underbrace{v}_{z}\rangle$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$. DO NOT SOLVE the integral.

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$



$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle$$

$$\mathbf{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle$$

$$\iint_D \langle v, u\sin v, u\cos v \rangle \cdot \langle \sin v, -\cos v, u \rangle\, dA$$

$$= \boxed{\int_0^1 \int_0^{\pi} \left(v\sin v - u\sin v\cos v + u^2\cos v \right) dv\, du}$$

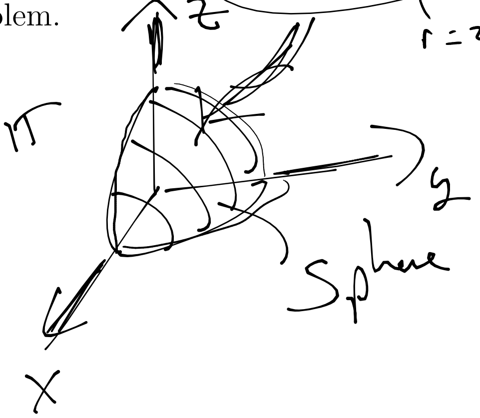
9) Setup the four integrals to find the center of mass of the portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant. Choose your own density function for this problem.

$\rho =$

$0 \le \phi \le \pi/2$

$0 \le \theta \le \pi/2$

$r = 2$



unit sphere

$$S \to 2\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle = \mathbf{r}$$

$$0 \le \phi \le \pi/2 \quad 0 \le \theta \le \pi/2$$

$$dS = |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, dA$$

$$\mathbf{r}_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$$

$$\mathbf{r}_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \underline{4\sin\phi}$$

(see Scalar 16.6 / 7)

$$M = \int_0^{\pi/2}\int_0^{\pi/2} 4\boxed{\rho}\sin\phi \, d\phi\, d\theta$$

$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_0^{\pi/2} 8\boxed{\rho}\sin^2\phi\cos\theta \, d\phi\, d\theta$$

$$\bar{y} = \frac{1}{M}\int_0^{\pi/2}\int_0^{\pi/2} 8\boxed{\rho}\sin^2\phi\sin\theta \, d\phi\, d\theta$$

$$\bar{z} = \frac{1}{M}\int_0^{\pi/2}\int_0^{\pi/2} 8\boxed{\rho}\sin\phi\cos\phi \, d\phi\, d\theta$$

10) In the heat flow example 6 on page 1172 of the textbook many steps are left out of the text. Do the entire example filling in all the extra work left to the student (taking partials, taking gradients, finding the normal, explaining all equalities, etc).

#1 Show ...

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

$\nabla u = \langle u_x, u_y, u_z \rangle$

$= C\langle 2x, 2y, 2z \rangle = 2C\langle x, y, z \rangle$

SOLUTION Taking the center of the ball to be at the origin, we have

Temp $\propto \left(\sqrt{x^2+y^2+z^2}\right)^2$

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

so $-K\nabla u = -2KC\langle x, y, z \rangle$

where C is the proportionality constant. Then the heat flow is

??

$$\text{\#1} \quad \mathbf{F}(x, y, z) = -K\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}) = -2KC\langle x, y, z \rangle = -2KC\,\mathbf{r}$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is ?

#2 Sphere $x^2+y^2+z^2=a^2$

$$\text{\#2} \quad \mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = \frac{1}{a}\langle x, y, z \rangle \qquad \mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \boxed{???}$$

and so

$$\text{\#3} \quad \mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

repeat pages 1167-1168 of text book to get $\mathbf{n} = \frac{1}{a}\mathbf{r}$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$. Therefore the rate of heat flow across S is

$$\text{\#4} \quad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \iint_S dS$$

$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3$$

$$\text{\#3} \quad -2KC\langle x, y, z \rangle \cdot \frac{1}{a}\langle x, y, z \rangle = -\frac{2KC}{a}(x^2+y^2+z^2) = -\frac{2KC}{a}a^2$$

$$\boxed{\mathbf{F} \cdot \mathbf{n} = -2aKC}$$

$$\text{\#4} \quad \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -2aKC \boxed{\iint_S dS}$$

just surface area of sphere

b/c $SA = \iint_S 1\, dS$

$= -2aKC(4\pi a^2)$

Surface Area of sphere

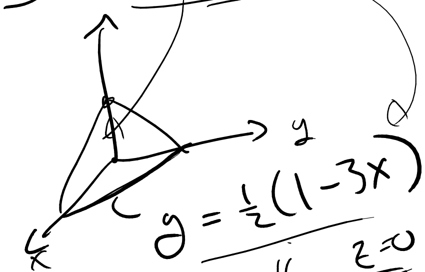
$= \boxed{-8KC\pi a^3}$

11) Do problem 8 on page 1179 of the textbook.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S}$$

S is $3x+2y+z=1$

so $z = 1-3x-2y$

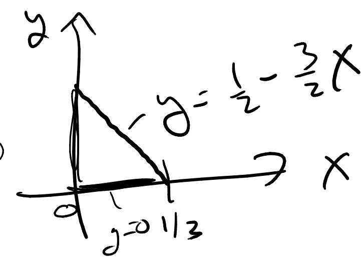


$y = \frac{1}{2}(1-3x)$

$z=0$

$$\mathbf{F} = \langle 1,\ x+yz,\ xy-\sqrt{z}\rangle$$

$$\operatorname{curl}(\mathbf{F}) = \nabla\times\mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z}\end{vmatrix} = \langle x-y,\ -y,\ 1\rangle$$



$y = \frac{1}{2} - \frac{3}{2}x$

$z = g(x,y)$

using $\boxed{16.7.10}$

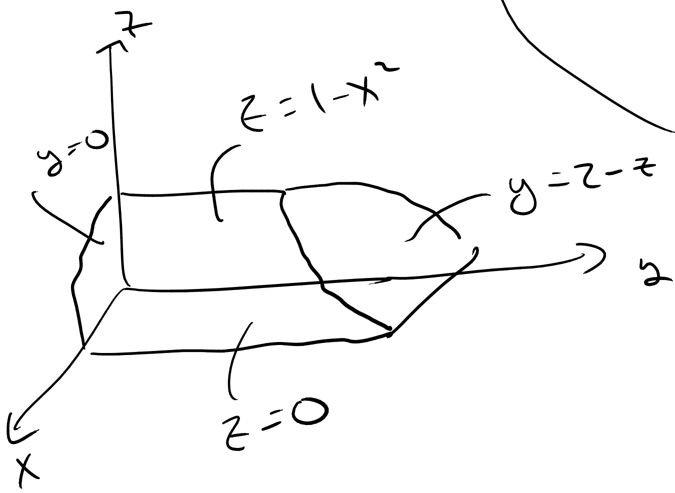
$$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^{1/3}\int_0^{\frac{1}{2}-\frac{3}{2}x} \big(-(x-y)(-3) - (-y)(-2) + 1\big)\,dy\,dx$$

$$= \int_0^{1/3}\int_0^{\frac{1}{2}-\frac{3}{2}x} (3x-5y+1)\,dy\,dx = \boxed{\frac{1}{24}}$$

use software

12) Redo the divergence theorem example 2 on page 1183 of the textbook using the the vector field $\langle \sin(yz), yz + e^z, z^2 + xy^3 \rangle$ rather than the one given in the textbook's example.

$\iint_S \mathbf{F} \cdot d\mathbf{S}$ of $\boxed{\mathbf{F} = ?}$ and S is region bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y = 2 - z$



$$\operatorname{div}(\mathbf{F}) = 0 + z + 2z = 3z$$

so $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3z \, dV$$

$$= 3 \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} z \; dy \, dz \, dx$$

$$= \boxed{\frac{16}{7}}$$

use software