MATH 530 ... EXAM 2

0) Exam Start Time:

0) Name: Key

0) MyWSUid:

0) Please explain all your answers in the exam.

a_n = # of {0,1,2} seq with no 1 to right of 2 length is n

start with 1

start w/ 0

$a_{n-1} + a_{n-1} + 2^{n-1}$ — start with 2

1) Explain the solution $a_n = 2a_{n-1} + 2^{n-1}$ to the problem of finding a recurrence relation for the number of n-digit ternary (0,1,2) sequence in which no 1 appears anywhere to the right of any 2.

$$a_n = 2a_{n-1} + \boxed{2^{n-1}}$$

(place 0 or 1) and (count rest of n-1 digits)

(place 2) and (there are two options (0 or 2) for rest n-1 digits)

$\quad = 1 \qquad = 2^{n-1}$

2 | | | ... | (n-1)

String of length n →

2) Find a recurrence relation for the number of ways of giving n dollars using 1 dollar coins, 1 dollar bills, 2 dollar bills, 10 dollar coins, 10 dollar bills, and 20 dollar bills.

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-10} + a_{n-10} + a_{n-20}$$

ANS → $a_n = 2a_{n-1} + a_{n-2} + \boxed{2a_{n-10}} + a_{n-20}$

2 options for \$1

extra for info

2 options for \$10

$a_0 = 1$

$a_1 =$

$\boxed{a_0 = 1}$

\$1 $a_1 = 2a_0 + \cancel{a_{-1} + 2a_{-9} + a_{-19}}$

$\vdots$

$a_{19} =$

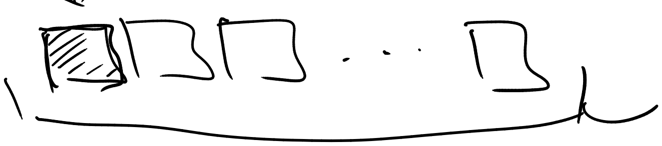
Note: for initial values you can use $a_n =$ [illegible] $n < 0$ and $a_0 = 1$

3) Find a recurrence relation for the numer of ways to arrange three types of blocks for a wall that is n feet long: green blocks (1 foot long), blue blocks (3 feet long), and red blocks (5 feet long) with the condition that there may be no subsequence of red, green, blue

$$a_n = \underbrace{a_{n-1}}_{\text{green and rest}} + \underbrace{a_{n-3}}_{\text{blue and } n-3 \text{ ft to go}} + \left(\underbrace{a_{n-5}}_{\text{red and } n-5 \text{ ft to go}} - \underbrace{a_{n-9}}_{n-9 \text{ ft to go}} \right)$$

subtract over count of rgb = 9ft

g? b? r?



n - feet

4) Solve and verify the solution to the recurrence relation $a_n = 3a_{n/3} + 2n$.

Use #3

table to know $\quad a_n = c\, a_{n/k} + f(n)$

① $a_n = a_{n/k} + d \xrightarrow{\text{Solve}} a_n = d\lceil \log_k n \rceil + A$

② $a_n = k\, a_{n/k} + d \rightarrow a_n = An - \frac{d}{k-1}$

③ $a_n = k\, a_{n/k} + dn \rightarrow a_n = dn(\lceil \log_k n \rceil + A)$

④ $a_n = c\, a_{n/k} + dn \rightarrow a_n = A n^{\log_k c} + \left(\frac{kd}{k-c}\right) n$, $c \neq k$

table

$$\boxed{a_n = 2n(\lceil \log_3 n \rceil + A)}$$

rec. relation $\quad a_n = 3a_{n/3} + 2n$

Verify: $2n(\lceil \log_3 n \rceil + A) \stackrel{?}{=} 3\left[\frac{2}{3}n\left(\lceil \log_3 \frac{n}{3} \rceil + A\right)\right] + 2n$

$\stackrel{?}{=} 2n\left(\lceil \log_3 \frac{n}{3} \rceil + A\right) + 2n$

$\lceil \log_3 n - \log_3 3 \rceil = \lceil \log_3 n \rceil - 1$

$\stackrel{?}{=} 2n(\lceil \log_3 n \rceil - 1 + A) + 2n$

$= 2n(\lceil \log_3 n \rceil + A)$

So true!

5) Solve the recurrence relation $a_n = a_{n-1} + 5a_{n-2} + 3a_{n-3}$, $a_0 = 2$, $a_1 = 1$, and $a_2 = 12$. Also give the first 5 values of the sequence.

$r^3 = r^2 + 5r + 3 \longrightarrow \boxed{r^3 - r^2 - 5r - 3 = 0}$ possible roots $\pm 3, \pm 1$

$$\begin{array}{r} r^2 - 2r - 3 \\ r+1 \,\overline{)\, r^3 - r^2 - 5r - 3} \\ \underline{r^3 + r^2} \\ -2r^2 - 5r \\ \underline{-2r^2 - 2r} \\ -3r - 3 \end{array}$$

$(r+1)(r^2 - 2r - 3) = 0$

$(r+1)(r+1)(r-3) = 0$

$r = -1 \quad r = -1 \quad r = 3$

so...

$$\boxed{a_n = A(3^n) + (B + Cn)(-1)^n}$$

$a_0 = 2 \longrightarrow 2 = A + B$

$a_1 = 1 \longrightarrow 1 = 3A - B - C$

$a_2 = 12 \longrightarrow 12 = 9A + B + 2C$

$\longrightarrow 3 = 4A - C$

$\longrightarrow 13 = 12A + C$

$16 = 16A$

$A = 1$

$B = 1$

$C = 1$

so $\boxed{a_n = 3^n + (1+n)(-1)^n}$

+8

+2

$a_0 = a_{n-1} + 5a_{n-2} + 3a_{n-3} \quad a_0 = 2, a_1 = 1, a_2 = 12$

a_0	a_1	a_2	a_3	a_4	a_5
2,	1,	12,	23,	86,	237, ...

$12 + 5(1) + 3(2) = 23$

$23 + 5(12) + 3(1) = 86$

$86 + 5(23) + 3(12) = 237$

vs

$3^3 + (1+3)(-1)^3 = 23$

$3^4 + (1+4)(-1)^4 = 86$

$\vdots$

6) Find and solve a recurrence relation for the number of ways to arrange flags on an n foot flagpole using three types of flags: red flags 2 feet high, yellow flags 2 feet high, and blue flags 1 foot high.

$$\boxed{a_n = \underbrace{1}_{\text{blue}} \cdot a_{n-1} + \underbrace{2}_{\text{y or r}} a_{n-2}}$$

$a_0 = 1$, $a_1 = 1$

$$r^2 = r + 2$$
$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$r = 2 \qquad r = -1$$

$$a_n = A \cdot 2^n + B \cdot (-1)^n$$

$$1 = A + B \qquad n=0$$
$$1 = 2A - B \qquad n=1$$

$$2 = 3A \qquad A = 2/3$$
$$B = 1/3$$

$$A = 2/3 \qquad B = 1/3$$

$$\boxed{a_n = \frac{2}{3} 2^n + \frac{1}{3}(-1)^n} = \frac{2^{n+1} + (-1)^n}{3}$$

7) Explain the solution $1 - \frac{10+1}{2^{10}}$ to the problem of finding the probability that at least two tails (not necessarily consecutive) will appear when a coin is flipped 10 times.

$|U| = 2^{10}$ $\quad P(\text{at least two Tails}) = \frac{2^{10} - (10+1)}{2^{10}} = 1 - \boxed{\frac{10+1}{2^{10}}}$

$\neg$(at least 2 tails)

$\neg$(at least 2) = 1 or 0 tails

$|\text{one tail}| = 10$ ← 1 tail for 10 spots

$|\text{zero tails}| = 1$ (all heads)

All $- \neg$(at least 2 tails)

$= 2^{10} - (10+1)$

$$P(E) = \frac{2^{10} - (10+1)}{2^{10}}$$

$$= \boxed{1} - \boxed{\frac{10+1}{2^{10}}}$$ ← prob $\neg$(at least 2)

100%

8) How many 20-digit numbers are there in which at least one pair of consecutive digits are the same?

10 9 9 ... 9

20 spots

$|U| = 10^{20}$ all 20 digit #'s

$\boxed{\text{so}}$ all $- \neg$(at least one pair)

$\neg$(at least one pair) = no pairs

$|\text{no pairs}| = 10 \cdot 9 \cdot 9 \cdot \ldots \cdot 9 = 10 \cdot 9^{19}$

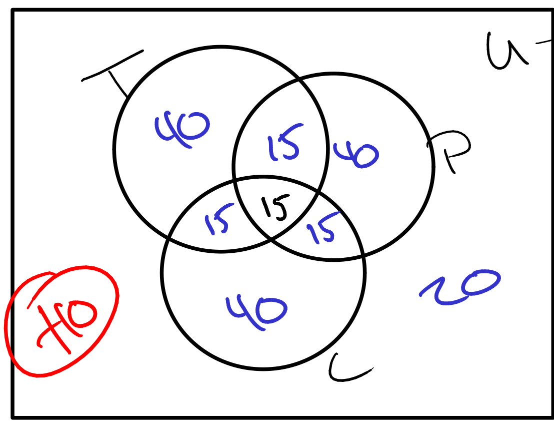
don't use 1st $\quad$ don't use 2nd $\quad$ don't use 3rd

$|\text{at least one pair}| = \boxed{10^{20} - 10 \cdot 9^{19}}$

(all $-$ no pairs) = at least one pair

8.1 #15

9) Draw a Venn Diagram and find the number of students in each of its 8 regions for the problem: A school has 200 students and the subjects of trigonometry, probability, and advanced counting happen to have 85 students each. There are 30 students taking any given pair of these subjects, and 15 students taking all three subjects.



$|U| = 200$

$|T| = 85$

$|P| = 85$

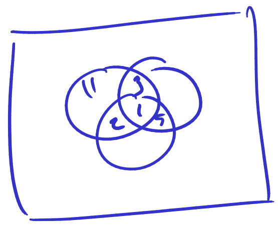
$|C| = 85$

$|T \cap P| = 30$

$|T \cap C| = 30$

$|P \cap C| = 30$

$|T \cap P \cap C| = 15$



$2+3+5 = 30$

ta

- no diag. soln
- no solns
- class of solns

$|\overline{A_1}\,\overline{A_2}\,\overline{A_3}\,\overline{A_4}\,\overline{A_5}| = |U| - S_1 + S_2 - S_3 + S_4 - S_5$

10) Explain the solution ...

$$6^4 - C(6,1) * 5^4 + C(6,2) * 4^4 - C(6,3) * 3^4 + C(6,4) * 2^4 - C(6,5) = 0$$

... for the problem of finding how many ways are there to roll four distinct dice so that all six faces appear.

$d_1 \; d_2 \; d_3 \; d_4$

$\boxed{ans = 0}$ (easy versa)

$|U| = 6^4$ all possible rolls

$S_1 = C(6,1)\,5^4$ all possible rolls missing 1 face

$S_2 = C(6,2)\,4^4$ " " " " 2 faces

$S_3 = C(6,3)\,3^4$ " " " " 3 faces

$S_4 = C(6,4)\,2^4$ " " " " 4 faces

$S_5 = C(6,5)\,1^4$ " " " " 5 faces

$(0, 2, 3, 4, \ldots, 9)$

11) How many n digit decimal sequences (using digits 0, 1, 2, ..., 9) in which digit 1 is not used, or digit 2 is not used, or 3 is not used?

$$|\boxed{A_1} \cup A_2 \cup A_3| = S_1 - S_2 + S_3$$

A_1: no 1, A_2: no 2, A_3: no 3

S_1: no 1 or no 2 or no 3

S_2: no 1 and no 2, no 1 and no 3, no 2 and no 3

S_3: no 1, no 2, no 3

$$= \left[\underbrace{9^n + 9^n + 9^n}_{S_1}\right] - \left[8^n + 8^n + 8^n\right] + \left[7^n\right]$$

$$= \boxed{3 \cdot 9^n - 3 \cdot 8^n + 7^n}$$

12) What is the probability to select a 6-card hand from a regular 52-card deck such that the hand contains at least one card in each suit?

$|U| = C(52, 6)$

A_i = does not have suit 'i'

$\bar{A}_i$ - has suit 'i'

$$|\bar{A}_1 \bar{A}_2 \bar{A}_3 \bar{A}_4| = |U| - S_1 + S_2 - S_3 + S_4$$

$S_1 = \binom{4}{1}\binom{52-13}{6} = 4\binom{39}{6}$ ← don't have 1 suit

$S_2 = \binom{4}{2}\binom{52-\boxed{26}}{6} = 6\binom{26}{6}$ ← don't have 2 suits

$S_3 = \binom{4}{3}\binom{52-39}{6} = 4\binom{13}{6}$ ← don't have 3

$S_4 = \binom{4}{4}\binom{52-52}{6} = 0$ ← don't have 4

$$P(E) = \frac{\binom{52}{6} - 4\binom{39}{6} + 6\binom{26}{6} - 4\binom{13}{6} + 0}{\binom{52}{6}}$$

0) What is the time you ended working on the exam and started scanning it?