Math 530

Graphs

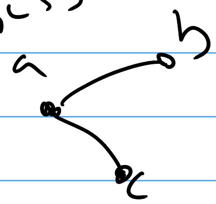
$G = (V, E)$

set of vertices — non-empty

set of edges — undirected $\{a,b\}$, Directed (a,b)

Representing Graphs

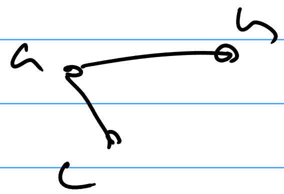
(1) lists $V = \{a,b,c\}$, $E = \{\{a,b\}, \{a,c\}\}$



(2) dictionary / directory

a : b, c
b : a
c : a

(3) Visual

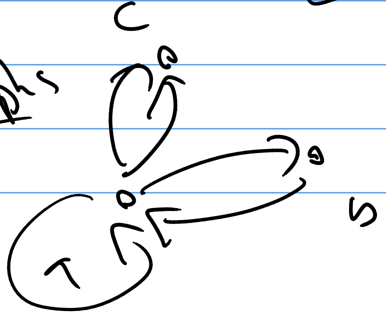


(4) Matrices — Adjacency Matrices

$$A_G = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

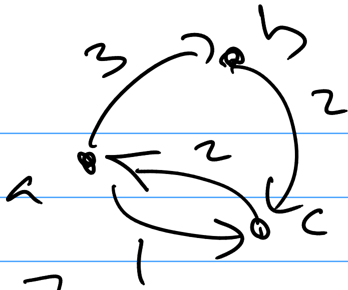
bit matrix 1 = edge
0 = no edge

dir Multigraphs



$$A_G = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Weighted graphs



$$A_G = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

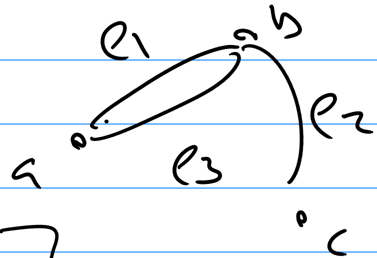
$0 \equiv$ no edge

$\# \equiv$ cost / weight

vs # is cost (and) prob is a minimal task.

$$A_G = \begin{bmatrix} \infty & 3 & 1 \\ \infty & \infty & 2 \\ 2 & \infty & \infty \end{bmatrix}$$

$\boxed{\text{Incidence Matrix}}$



$$I_G = \begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

(a) adj. Matrix

typical use of this is $I_G \cdot I_G^T = A_G$

vertex / vertex

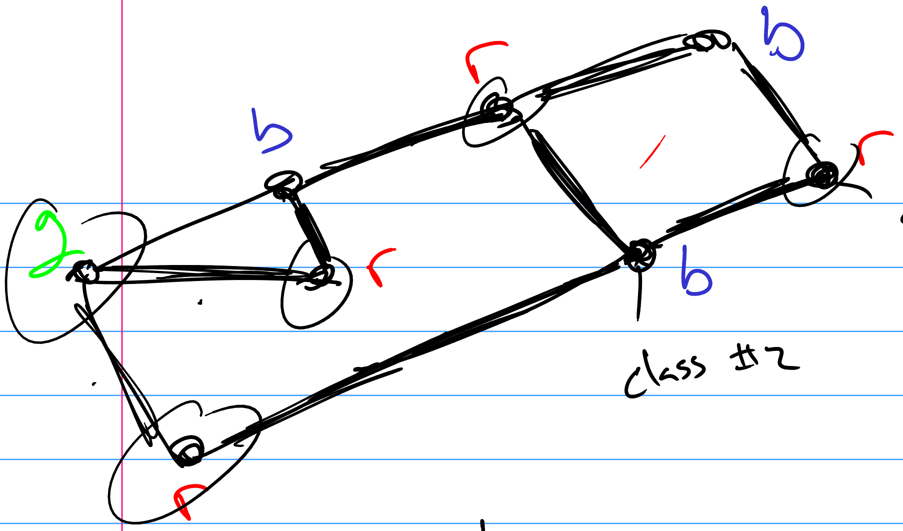
or

$I_G^T \cdot I_G = A_G$

edge / edge

Why Graphs?

ex. "One goal of graph theory is to find useful relationships between seemingly unrelated graph concepts"



V = classes

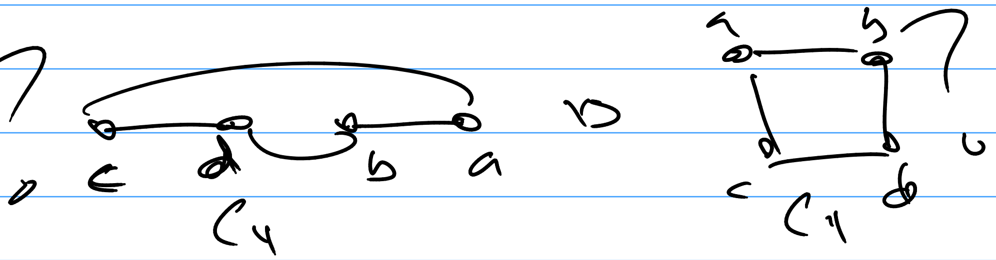
class #1

E = ?
any common student/person exists

- coloring
- edge covers
- independent sets

sets of vertices
no edges between a set's elements
ex: reds or blues

Same?



Isomorphic

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are $\boxed{\text{isomorphic}}$

there must be a bijection from G_1's vertices to G_2's vertices that preserves edges.

If a bijection does exist we call it an isomorphism and there are obvious things that are true

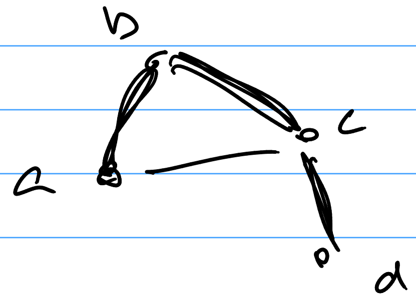
Invariants

1. $|V_1| = |V_2|$
2. $|E_1| = |E_2|$
3. degree's have to match. and map same degree vertices to same degree vertices
4. Neighborhood of v is the set of its adj. vertices

and neighborhoods are preserved.

5) paths are preserved.

Def: path is a seq. of edges.



$\{a,b\},\{b,c\},\{c,d\}$ length = 3

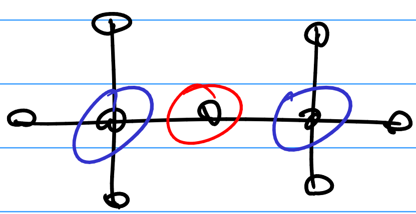
a, b, c, d

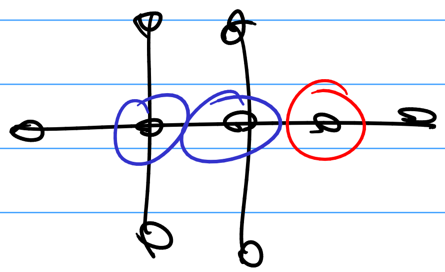
Def: path starts / ends @ same vertex → $\boxed{\text{circuit}}$

Def: if edges in path only occur once → $\boxed{\text{simple}}$

$\boxed{\text{Broken invariant}}$ → no isomorphism

and G_1, G_2 are $\boxed{\text{not}}$ isomorphic





both → $|V| = 9$

→ $|E| = 8$

$\deg(v) = 1$

$\deg(v) = 2$

$\deg(v) = 4$

how many?

6

1

2

Not isomorphic

to show $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are isomorphic...

you need the isomorphism

$$A_{G_1} = \begin{array}{c|c} & v_1 \ v_2 \ \cdots \ v_n \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} & \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix} \end{array}$$

f

$v_1 \to f(v_1) \in V_2$

$v_2 \to f(v_2) \in V_2$

$\vdots$

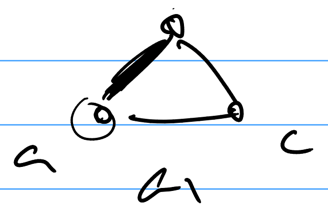
$v_n \to f(v_n) \in V_2$

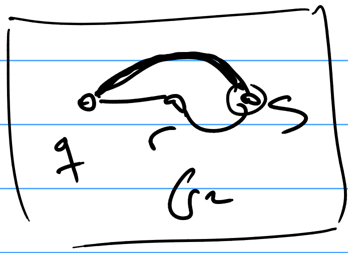
$$A_{G_2} = \begin{array}{c|c} & f(v_1) \ \cdots \ f(v_n) \\ \begin{array}{c} f(v_1) \\ \vdots \\ f(v_n) \end{array} & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array}$$

$V_1 = \{v_1, v_2, \dots, v_n\}$

ex

$$A_{G_1} = \begin{array}{c|c} & a \ b \ c \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \qquad A_{G_2} = \begin{array}{c|c} & s \ q \ r \\ \begin{array}{c} s \\ q \\ r \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$





f

$a \to s$

$b \to q$

$c \to r$

$A_{G_1} = A_{G_2}$ in isomorphism order

$\Rightarrow$ isomorphic

1.3 Edge Counting

Th^m $\sum_{v \in V} \deg(v) = 2|E|$

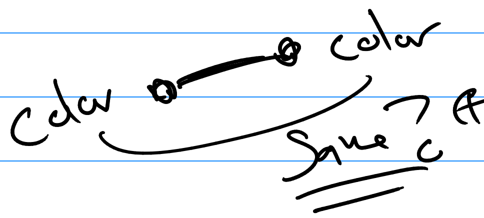
Cor. even number of odd degree vertices

$\boxed{\text{Thm}}$ G is bipartite $(V = P_1 \cup P_2)$

Partition V into 2 subsets that are independent sets.

(a) <u>coloring thm</u>

use two colors to color vertices <u>and</u> we never have



(b) <u>circuits</u> all have even edges