Math 322

asymmetry $\equiv$ irreflexive $\wedge$ antisym.

irreflexive: no loops

or

$M_R = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$

antisym.: if $a \to b$ then $a \leftarrow b$ no back edge

or

$M_R = \begin{bmatrix} & 0 & 1 \\ 1 & & 0 \\ 0 & 0 & \end{bmatrix}$

asymmetric $M_R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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all properties

		digraph	matrix
Reflexive	$\forall e (eRe)$	e — all e have loops	$M_R = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$
Irreflexive	$\forall e (e \not R e)$	$\bullet e$ — no loops	$M_R = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}$
Symmetric	$\forall a \forall b (aRb \to bRa)$	$a \rightleftarrows b$ — any edge has it's "pair"	$M_R = \begin{bmatrix} & 1 & 0 \\ 1 & & \\ 0 & & \end{bmatrix}$
antisymmetric	$\forall a \forall b (aRb \wedge bRa \to a = b)$ or $\equiv \forall a \forall b (a \neq b \to \neg (aRb \wedge bRa))$	digraph: no paired edges	$M_R = \begin{bmatrix} & 0 & 1 \\ 1 & & 0 \\ 0 & 0 & \end{bmatrix}$

a symmetric (See above)

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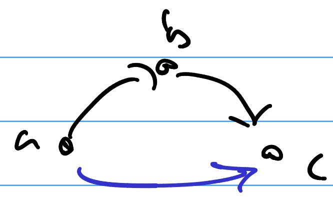
transitive: $\forall a \forall b \forall c \, (aRb \wedge bRc \rightarrow aRc)$

digraph: 

must have trans edge

Matrix ?

no visual inspection

Operations on Relations

Union $R_1 \cup R_2 = \{ (a,b) \mid aR_1b \vee aR_2b \}$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

Intersection $R_1 \cap R_2 = \{ (a,b) \mid aR_1b \wedge aR_2b \}$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

Composition 

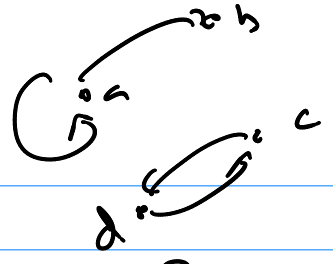
$$R_2(R_1(a)) = c$$

$$(R_2 \circ R_1)(a) = c$$

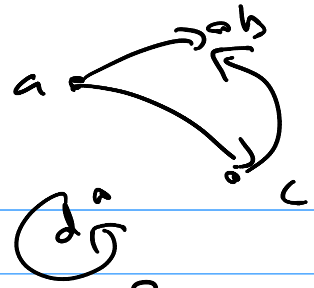
2nd relation   1st relation

$$M_{R_2 \circ R_1} = M_{R_1} \odot M_{R_2}$$

ex



R_1



R_2

$$M_{R_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R_2} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Transitive?

Th^m R is transitive if and only if $\forall n \; R^n \subseteq R$, $n = 1, 2, 3, \ldots$

$R \subseteq R$

$R \circ R \subseteq R$

$R \circ R \circ R \subseteq R$

$\vdots$

where: $R^n = \underbrace{R \circ R \circ \cdots \circ R}_{n\text{-times}}$

Pf

case 1 (R is trans) $\longrightarrow$ ($\forall n \; R^n \subseteq R$)

case 2 ($\forall n \; R^n \subseteq R$) $\rightarrow$ (R is trans.)

$R^n = R \circ R \circ \cdots$

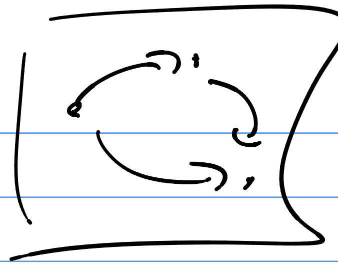
$R \circ R$

Idea for case: direct: assume R is trans.

then $\forall n \; R^n \subseteq R$, $\boxed{n=1}, 2, 3, \ldots$

Induction!

case 2 idea $\boxed{\forall n\ R^n \subseteq R} \rightarrow R$ is trans



direct: assume $\forall n\ R^n \subseteq R$

by univ. inst., $\boxed{R^2 \subseteq R}$ is true!

but $R^2 = R \circ R$ and composition is $aRb \wedge bRc$

so $R^2 \subseteq R$ means ... $\boxed{a\, R \circ R\, c} \rightarrow aRc$

what is subset?

$aRb \wedge bRc \rightarrow aRc$

Why?

① Equivalence relations.

If a relation, R, is reflexive, symmetric, and transitive we call R an equivalence relation.

(uses: checking if e_i "same as" e_j)

ex R on integers. $R = \{ (a,b) \mid a \equiv b \bmod 3 \}$
$= \{ (a,b) \mid a \equiv_3 b \}$

ex $4 \equiv_3 1 \equiv_3 7 \equiv_3 10 \equiv_3 -2$ etc

is R an equiv. relation?

$R = \{(a,b) \mid a \underset{3}{\equiv} b\}$

check: is R reflexive? symmetric? transitive?

$\forall e (eRe) \equiv \forall e (e \underset{3}{\equiv} e)$

is $3 \mid (e-e)$, $3 \mid 0$ yes!

symm: $\forall a \forall b (aRb \to bRa)$

so $a \underset{3}{\equiv} b \to b \underset{3}{\equiv} a$?