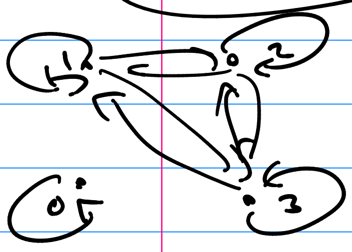
Math 322

$[e]_R = \{ s \mid e \sim s \}$ — R as an equiv. relation

6.3 #7



7. **Equivalence Classes.** Let $A = \{0,1,2,3\}$ and let

$$r = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1),(3,2),(2,3),(3,1),(1,3)\}$$

(a) Verify that r is an equivalence relation on A. (ref, sym, trans) $\boxed{\text{yes}}$

(b) Let $a \in A$ and define $c(a) = \{b \in A \mid arb\}$. $c(a)$ is called the **equivalence class of** a **under** r. Find $c(a)$ for each element $a \in A$. $[e]_R = c(e)$

(c) Show that $\{c(a) \mid a \in A\}$ forms a partition of A for this set A.

(d) Let r be an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes under r constitutes a partition of A.

$c(1) = [1]_R = \{1,2,3\} = [2] = [3] = c(2) = c(3)$

$c(0) = [0]_R = \{0\}$

Partition? $\boxed{\text{Yes}}$

$A = \{0, 1, 2, 3\}$

partition of a set are disjoint non-empty subsets such that their union is the entire set.

ex $A = \{a, b, c, d, e, f, g\}$

Partition of A: $S_1 = \{a\}$ $S_2 = \{b, d, g\}$ $S_3 = \{c, e, f\}$

Idea for (d)

(try contradiction)

Assume: R is an equiv. relation with equiv classes and the classes do not form a partition.

$\Rightarrow$ union $\neq$ set (missing elements)

but by ref. all elements $e R e$ so they are in an equiv. class.

Contradiction

b) Not - disjoint.

$$[e_i] \cap [e_2] \neq \phi$$

<u>and</u> $e_i \in [e_i] \wedge e_i \notin [e_2]$

ex $[e_i] = \{1, 2, 3, 4\}$

$[e_2] = \{4, 5, 6, 7\}$ <u>Contradiction</u>

Transitive Closure

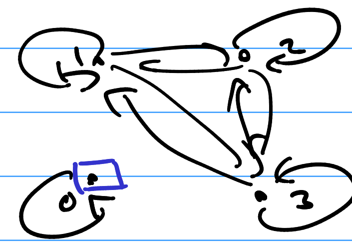
Warshal's Algorithm

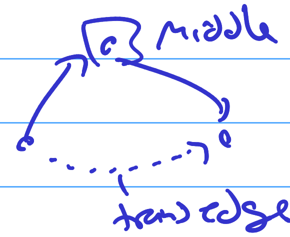
→ pick each vertex from v_1 to v_k

→ for each check

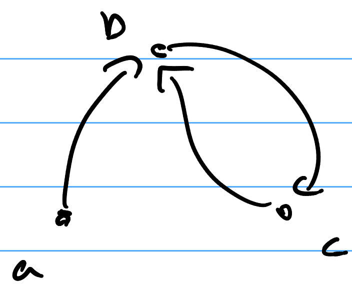
a) is it in a middle

<u>yes?</u> make sure to have trans. edge



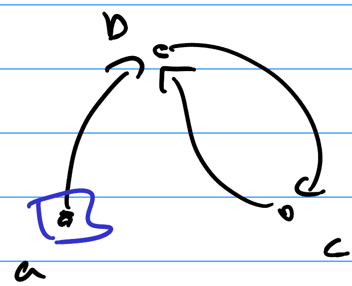


ex



$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 1

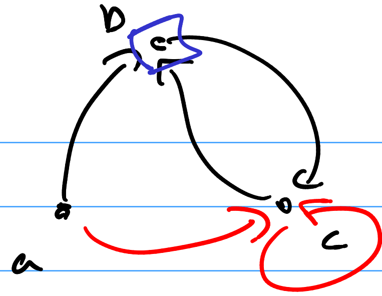


$$W_1 = \begin{array}{c|c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \boxed{1} & 0 \end{bmatrix} \end{array}$$

go from a

go to a

Step 2

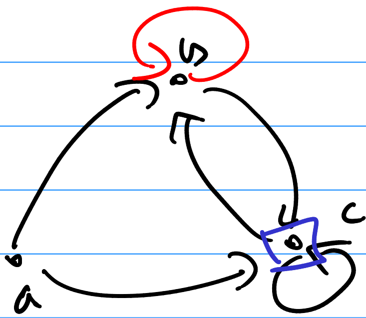


$$W_2 = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 0 & 0 & 1 \\ c & 0 & 1 & 1 \end{array}$$

b goes to $\square$

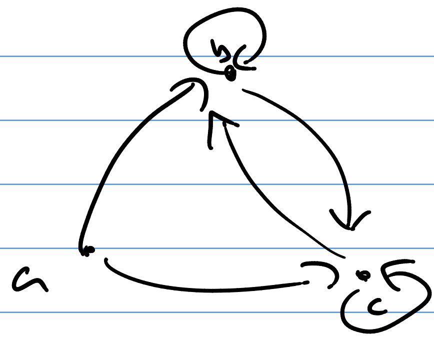
$\square$ go to b

Step 3



$$W_3 = \begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 0 & 1 & 1 \end{array}$$

R^*



$$M_{R^*} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Compare to

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]}$$

$$M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = ?$$

$$M_R^{[3]} = M_R^{[2]} \odot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = ?$$

Relations:

1. Properties
2. Equiv Relations (classes)
3. Posets (Hasse Diagrams)
4. Closures

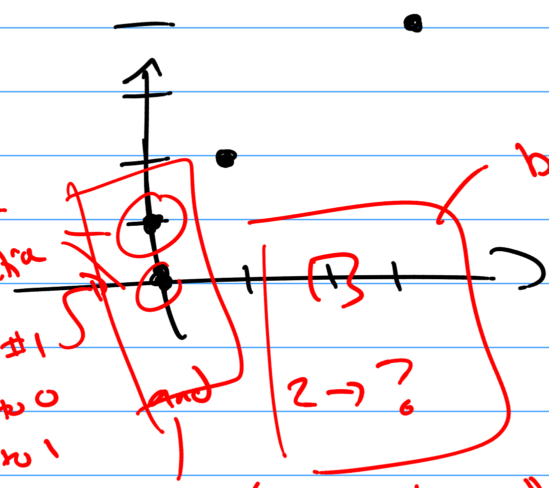
Special Relation: Function

given R a relation from set A to set B.
If every $a \in A$ is related to exactly one $b \in B$. R is called a function.

ex R on $\{0,1,2,3,4\}$

$$R = \{(0,0), (0,1), (1,2), (3,4)\}$$

Is this a function?



Not a function b/c #1 0 to 0 and 0 to 1

broken "every" one goes rule

$2 \to ?$

breaks "to exactly one" rule.