

# Math 322

Hw: Due Next Wed

6.1 (2,3)

6.2 (2,3,6)

6.3 (1,2,3,5,6,7,10)

~~ADS  
Textbook~~

Due Monday Feb 10

6.4 (1,3,4,5,7)

6.5 (1,2,4)

## R on set A

### Properties

(1) Reflexive  $\forall a (aRa)$

(2) Irreflexive  $\forall a (\neg aRa)$

(3) Sym  $\forall a \forall b (aRb \rightarrow bRa)$

(4) Anti-Sym  $\forall a \forall b (aRb \wedge bRa \rightarrow a=b)$

(5) Asym  $\forall a \forall b (aRb \rightarrow \neg bRa)$

(6) transitive  $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

Equiv. Relation:  $R$  on set  $A$  is an equivalence relation if it is ref, sym, and trans.

(\*) (Congruence is a classic example)  
 $R$  on  $\mathbb{Z}$ ,  $R = \{(a,b) \mid$

$$a \equiv_m b \iff \begin{cases} 1 & a \equiv b \\ 2 & b \equiv a \\ 3 & a \equiv b \text{ mod } m \\ 4 & a \equiv a \text{ mod } m \end{cases}$$

Show: ref? sym? trans?

Pract: Ref?  $aRa$ : " $a \equiv_a a$ "      ③  $a \equiv b \iff a - b \in \mathbb{Z}$   
 $b \equiv_m (a + m)$ ,  $m \neq 0$  is true

Sym?  $aRb \rightarrow bRa$  means  $a \equiv_m b \rightarrow b \equiv_m a$

$a \equiv_m b \rightarrow m \mid (b-a)$   $m$  is a factor of  $b-a$   
 then  $m$  is also a factor of  $-(b-a)$   
 so  $m$  is a factor of  $a-b$   
 so  $m \mid a-b \wedge b \equiv_m a$

trans?  $aRb \wedge bRc \rightarrow aRc$   
 means  $m \mid (b-a) \wedge m \mid (c-b) \rightarrow m \mid (c-a)$

$\hookrightarrow m \mid (b-a) + (c-b)$   
 $\rightarrow m \mid c-a \rightarrow a \equiv_m c$  true!

uses of equivalence relations  $\rightarrow$  equivalent classes

equivalence class,  $[e]_R$  when to use this subscript? is all elements  
 equivalent to our representative  $e$ .

$$[e]_R = \{ s \mid e R s \}$$

(Ex)  $R_1$  on  $\mathbb{Z}$  int  $R_1 = \{ (a,b) \mid a \equiv_4 b \}$

$R_2$  on  $\mathbb{Z}$  int  $R_2 = \{ (a,b) \mid a \equiv_5 b \}$

$$[0]_{R_1} = \{ \dots, -8, -4, 0, 4, 8, 12, \dots \}$$

$\forall c \quad R_1$  is an equiv relation each of these  
 representatives are the "Same as" any other.

$$\begin{aligned} [0]_R &= \{-, -2, -4, 0, 4, 8, \dots\} \\ [-1]_R &= \{-, -7, -3, 1, 5, 7, \dots\} \\ [2]_R &= \{-, -6, -2, 2, 6, 10, \dots\} \end{aligned}$$

Ex

$$\begin{aligned} [3]_R &= \{-, -5, (-1) \cdot 3, 7, 11, \dots\} \\ &= (5^{101, 232} + 4^{10^{10^{10}}}) \bmod 4 \\ &= (1^{101, 232} + 0^{10^{10^{10}}}) \bmod 4 = 1 \end{aligned}$$

Note: the congruence classes of  $R$  on  $A$  will partition set  $A$ .

$$\textcircled{1} \quad A = [e_1] \cup [e_2] \cup \dots \cup [e_k]$$

$$\textcircled{2} \quad [e_i] \cap [e_j] = \emptyset \text{ if } i \neq j$$

$$\textcircled{3} \quad \text{if } [e_i] \cap [e_j] \neq \emptyset \rightarrow [e_i] = [e_j]$$

### Partial Orderings

$R$  on set  $A$  is a partial ordering of  $A$  if it is ref, antisym, and transitive.

Ex  $R$  on  $\{2, 3, 4, 6, 8, 10, 12\}$

$$R = \{(a, b) \mid a \mid b\} = \{(a, b) \mid a \text{ is a factor of } b\}$$

a) Ref?  $aRa$  means  $a \mid a$  true.

b) antisym?  $aRb \wedge bRc \rightarrow a=b$

means  $(a \mid b \wedge b \mid a \rightarrow a=b)$

$$\begin{aligned} &\uparrow a \cdot k_1 = b \wedge b \cdot k_2 = a \\ &\text{gives } a \cdot k_1 \cdot k_2 = a \\ &\text{so } k_1 \cdot k_2 = 1 \end{aligned}$$

$$\text{so } a = b$$

true!

c) trans?  $aRb \wedge bRc \rightarrow aRc$   
means  $a/b \wedge b/c \rightarrow a/c$  true!

(ex)  $A = \{2, 3, 4, 6, 8, 0, 12\}$   $(2, 12) \in R$

Compare 2 to 12

$b/c$   $2/12 \rightsquigarrow 2 \nmid 12 \leftarrow$

2 to 3

$b/c$   $3/12 \rightsquigarrow 3 \nmid 12 \leftarrow$

8 to 4

$b/c$   $4/8 \rightsquigarrow 4 \nmid 8$

problem..

2 to 3 ?

$2 \nmid 3 \wedge 3 \nmid 2$

2, 3 are incomparable!

Notation: Posets are sets,  $S$ , with relations  
that are ref, anti sym, and transitive

$\rightsquigarrow$  a pair  $(S, \rightsquigarrow)$

symbol to mean any relation  
that is ref, anti sym, trans

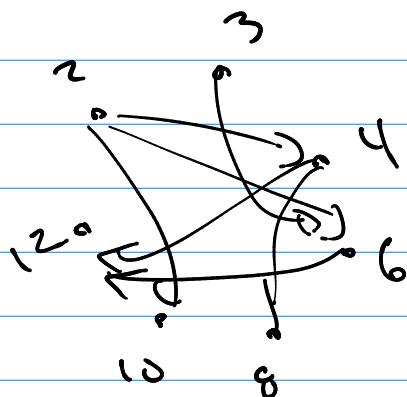
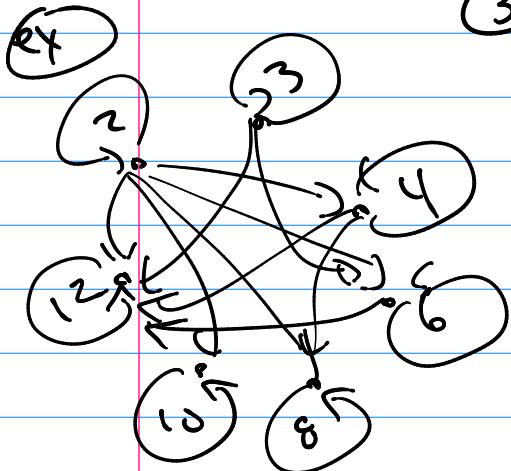
Visualize a Poset  $\rightarrow$  Hasse Diagram

$\rightarrow$  take digraph and simplify it by

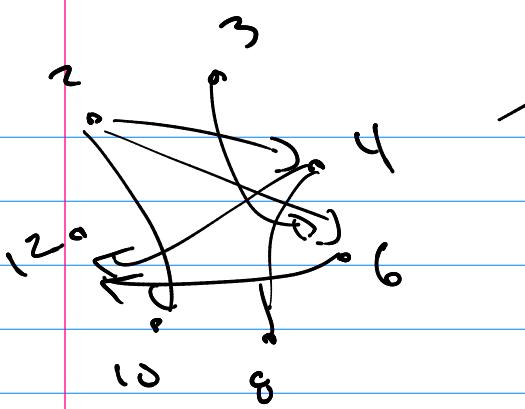
① remove loops (they are there, don't draw)

② remove trans edges ("")

③ remove arrows by making them all  
point up.

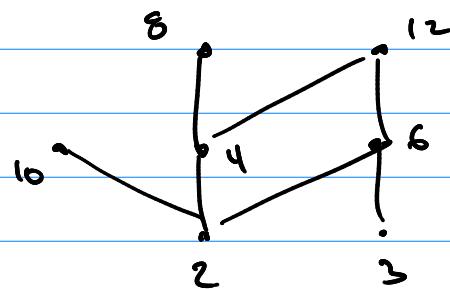


hide loops, hide tails



arrows  
up

Hausse Diagram

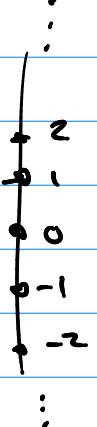


## Note on Partial Orderings.

- ① if every element compares to every other other element then the relation is a total order

Ex:

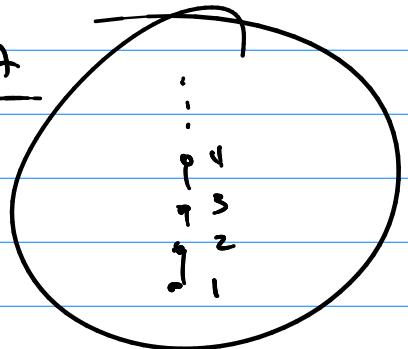
( $\mathbb{Z}, \leq$ )



- ② if you have a total order then any subset has a least element  $\rightarrow$  Well Ordered

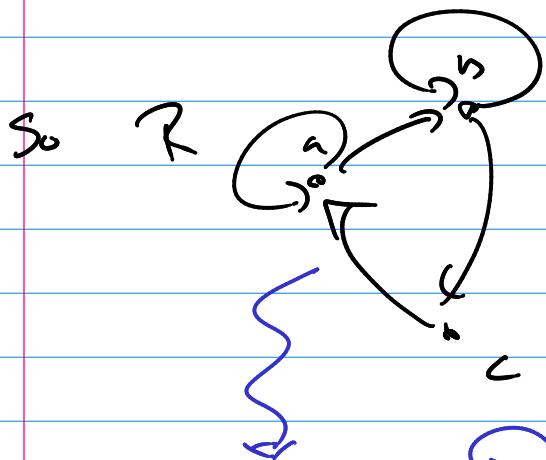
Ex:

( $\mathbb{Z}^+, \leq$ )



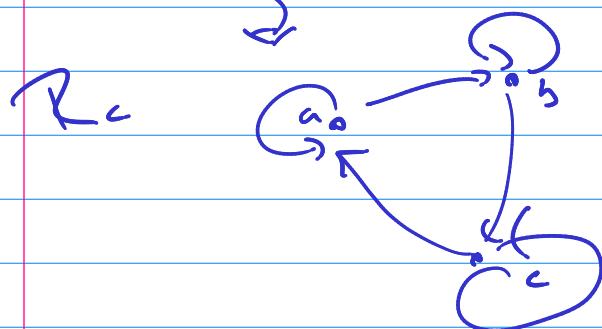
## Closures

What if  $R$  is not reflexive?  
 or not symmetric?  
 or not transitive?



reflexive? no b/c  $\text{cyclic} \rightarrow$  not here

$\boxed{PQ}$  minimal number of edges  
 to add to get a new relation that  $\boxed{PQ}$  reflexive



call the the reflexive closure of  $R$ .

To find the reflexive closure

$$M_R \vee I$$

(ex)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

## Symmetric Closure

$$[M_R \vee M_R^T]$$

