NAME:

MATH 511 - EXAM 1

Read

0) Time you started the exam (Note: Exam must be uploaded within two hours of downloading it):

0) Write your MyWSU ID and sign your name if you will abide by the rules of Academic Honesty:

1) Solve the system of equations. DO NOT use matrices. If your MyWSU ID starts with any letter from a to l let (a,b,c,d) = (0, 4, 4, 1). And if your mywsu ID starts with any letter from m to z let (a,b,c,d) = (0, 5, 5, 5).

$$\begin{aligned} x_4 - x_2 &= a \quad 0 \quad 0 \rightarrow \quad x_2 = x_4 \\ x_1 + x_2 - x_3 + x_4 &= b \quad 4 \quad 5 \\ 2x_1 + 2x_2 - x_4 &= c \quad 4 \quad 5 \\ 3x_1 - x_4 &= d \quad 1 \quad 5 \end{aligned}$$

$x_1 + 2x_2 - x_3 = 4$

$2x_1 + x_2 = 4$

$3x_1 - x_2 = 1$

$\oplus$

$5x_1 = 5 \qquad x_1 = 1$

$x_2 = 2, \quad x_4 = 2$

$1 + 4 - x_3 = 4$

$x_3 = 1$

$(x_1, x_2, x_3, x_4) = (1, 2, 1, 2)$

$x_1 + 2x_2 - x_3 = 5$

$2x_1 + x_2 = 5$

$3x_1 - x_2 = 5$

$5x_1 = 10 \qquad x_1 = 2$

$x_2 = 1, \quad x_4 = 1$

$2 + 2 - x_3 = 5$

$x_3 = -1$

$(x_1, x_2, x_3, x_4) = (2, 1, -1, 1)$

2) Solve the system of equations. Use Gaussian Elimination on an augmented matrix. If your MyWSU ID starts with any letter from a to l let (a,b,c,d) = (0, 4, 4, 1). And if your mywsu ID starts with any letter from m to z let (a,b,c,d) = (0, 5, 5, 5).

$$\begin{aligned} x_4 - x_2 &= a &\leftarrow \text{row } 2 \\ x_1 + x_2 - x_3 + x_4 &= b &\leftarrow \text{row } 1 \\ 2x_1 + 2x_2 - x_4 &= c &\leftarrow \text{row } 3 \\ 3x_1 - x_4 &= d &\leftarrow \text{row } 4 \end{aligned}$$

$r_3 + (-2)r_1 = r_3$

$r_4 + (-3)r_1 = r_4$

$$\left[\begin{array}{cccc|cc} 1 & 1 & -1 & 1 & 4 & 5 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 4 & 5 \\ 3 & 0 & 0 & -1 & 1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cc} 1 & 1 & -1 & 1 & 4 & 5 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & -4 & -5 \\ 0 & -3 & 3 & -4 & -11 & -10 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cc} 1 & 1 & -1 & 1 & 4 & 5 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & -4 & -5 \\ 0 & 0 & 3 & -7 & -11 & -10 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cc} 1 & 1 & -1 & 1 & 4 & 5 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & -4 & -5 \\ 0 & 0 & 0 & -5 & -10 & -5 \end{array}\right]$$

$r_4 + (-3)r_2 = r_4$

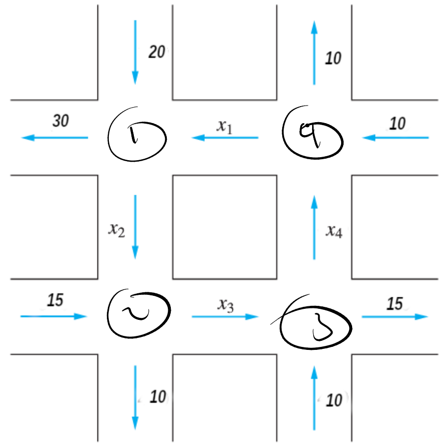
$-3r_2 + 2r_3 = Nr_3$

$$\rightarrow \left[\begin{array}{cccc|cc} 1 & 1 & -1 & 1 & 4 & 5 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & -4 & -5 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right] \rightarrow$$

$x_4 = 2$	$x_4 = 1$
$x_3 = 1$	$x_3 = -1$
$x_2 = 2$	$x_2 = 1$
$x_1 = 1$	$x_1 = 2$
$(1, 2, 1, 2)$	$(2, 1, -1, 1)$

Same as #1

3) Determine the values of x_i for the traffic flow diagram by using Gausian Elimination on an augmented matrix.



(1) $x_1 + 20 = x_2 + 30 \rightarrow x_1 - x_2 = 10$

(2) $x_2 + 15 = x_3 + 10 \rightarrow x_2 - x_3 = -5$

(3) $x_3 + 10 = x_4 + 15 \rightarrow x_3 - x_4 = 5$

(4) $x_4 + 10 = x_1 + 10 \rightarrow x_4 - x_1 = 0$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 5 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & -1 & 0 & 1 & 10 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & -1 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 & -5 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 10 \end{array}\right]$$

no solution!

4) Perform the indicated operations.

a) $\begin{pmatrix} -1 & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}^T - 2 \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

can not do ops

2×3 $\quad (3\times 2)$ or $(2\times 3)(2\times 3)$

b) $\begin{pmatrix} a & b & c \end{pmatrix}^T \begin{pmatrix} x & y & z \end{pmatrix}$

$$\underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} x & y & z \end{bmatrix}}_{1\times 3} = \underbrace{\begin{bmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{bmatrix}}_{3\times 3}$$

b) $\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix}^T$

$$\underbrace{\begin{pmatrix} a & b & c \end{pmatrix}}_{1\times 3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3\times 1} = ax + by + cz$$

5) Calculate $A^2 - 3I$ for the matrix A ...

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & -2 \end{bmatrix}$$

6) Solve $XA + B = I + 2C$ for matrix X and then find X using the below matrices ...

$XA\,A^{-1}\,(I+2C-B)A^{-1}$ $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

$$X = \boxed{(I+2C-B)A^{-1}}$$

A^{-1} is

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad A^{-1}$$

$$X = \left(\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} + \underbrace{\begin{pmatrix} 4 & 2 & 2 \\ -4 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}}_{2C} - \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{-B}\right) \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 4 & 0 & 3 \\ -6 & 6 & 0 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 4 & -8 & 7 \\ -6 & 18 & -18 \\ 2 & -3 & 0 \end{pmatrix}}$$

7) Find the LU factorization for the given matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$$

$E_3 E_2 E_1 A = U$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ -1 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_2 + (-2) r_1 = r_2$

$r_3 + (1) r_1 = r_3$

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$r_3 + (1) r_2 = r_3$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

check!

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} U$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} U$$

$$A = LU$$

8) Find A^{-1} for the the given matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 5 \\ -1 & 0 & -2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & -3 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$A^{-1} = \boxed{\begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}}$$

If you want check: $A^{-1}A = I$

$AA^{-1} = I$

9) Given matrix A

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

a) Find $\det(A)$ by co-factors.

$$+3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 3(1) - 2(-4) = \boxed{11}$$

b) Find $\det(A)$ by elimination.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -4 & -9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -11/3 \end{vmatrix} = \boxed{11}$$

$$r_4 + \left(-\frac{4}{3}\right) r_3 = r_4$$

c) Label matrix A using the words 'singular', 'non-singular', 'invertable', and/or 'non-invertable' as appropriate. And discuss what would happen if you had the system of linear equations $A\mathbf{x} = \mathbf{0}$ to solve.

$\det(A) = 11$ so A is non-singular

$11 \neq 0$

A^{-1} is invertable

A^{-1} exists

homogeneous

solve $\boxed{A\mathbf{x} = \mathbf{0}}$ then only trivial soln.

10) Given matrix A

3X3

$$A = \begin{pmatrix} 1 & 1 & -c \\ 0 & a-b & a \\ 1 & 1 & -a \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & -c \\ 0 & a-b & a \\ 1 & 1 & -a \end{vmatrix} = \begin{vmatrix} 1 & 1 & -c \\ 0 & a-b & a \\ 0 & 0 & a-c \end{vmatrix}$$

a) What conditions must the scalars a, b, and c satisfy for A to be singular?

$\det(A) = 0$

$= 1(a-b)(a-c)$

$\det(A) = 0$ or $(a-b)\cdot(a-c) = 0$

so ... $a-b=0$ or $a-c=0$

$\boxed{a=b \text{ or } a=c}$

b) What conditions for A to be non-singular?

$\det(A) \neq 0$ so ... not $a=b$ or $a=c$

$a \neq b$ and $a \neq c$

c) Give an example matrix A by choosing a, b, and c so that A is invertable.

$a=1$, $b=2$, $c=2$

$A = [\,?\,]$

11) a) Let A be a 4 x 4 matrix, what are $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, and $\mathbf{a_4}$ in relation to matrix A?

$A = [a_1 \; a_2 \; a_3 \; a_4]$

$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 = 0$

A's cols.

b) If $\mathbf{a_1} + \mathbf{a_3} = \mathbf{a_2} + \mathbf{a_4}$, then how many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain.

$a_1 - a_2 + a_3 - a_4 = 0$

$A \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0$

non-trivial solution!

soln

c) Would A be invertable? Explain.

not-invertible.

d) Is A singular or non-singular? Explain.

0) Time you ended the exam and started to upload: