Math 511

3.2 Spans // Subspaces

∘) Subset of a Vectorspace

Subspace?

① $0 \in S$?

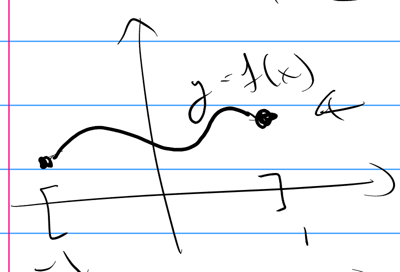
② $\alpha v \in S$?

③ $v_1 + v_2 \in S$?

Pre-work

ex) 3.2 #6 Vector Space is $C[-1,1]$

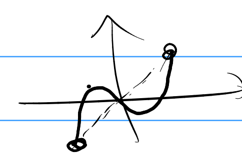
all cont. functions $y=f(x)$ over $[-1,1]$



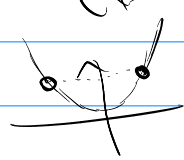
6b) Subset is odd functions

a) $f(-x) = -f(x)$

b) symmetric about origin



even function

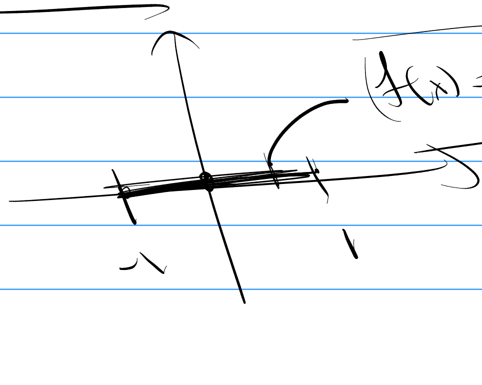


a) $f(-x) = f(x)$

b) sym about y-axis

What is 0? for $C[-1,1]$ 0 is $y=f(x)$

$y=0$



$f(x)=0$ is 0 of $C[-1,1]$

$g(x) + f(x) = g(x) + 0 = g(x)$

is S a subspace? $0 \in C[-1,1]$

1) is $f(x)=0$ a odd function? yes

$f(-x)=0$ so $f(-x)=f(x)$

$-f(x)=-0=0$ $\boxed{\text{yes}}$

Note: $f(-x)=0$ $f(x)=0$ $f(-x)=f(x)$

says it's even?!

2) if $g(x)$ is odd is $\boxed{\alpha g(x) \text{ odd?}}$

$\alpha g(-x) = \alpha\left(g(-x)\right) \overset{g \text{ is odd}}{=} \alpha\left(-g(x)\right)$

$= -\alpha\left(g(x)\right)$ $\boxed{\text{it is odd!}}$

3) if $g(x), h(x)$ are odd

show $(g+h)(x)$ is also odd.

$(g+h)(-x) = g(-x)+h(-x)$ $\boxed{\text{it's odd}}$

$= -g(x)-h(x) = -(g+h)(x)$

$\boxed{\text{So}}$ odd functions are a subspace of $C[-1,1]$

Spans

$\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$

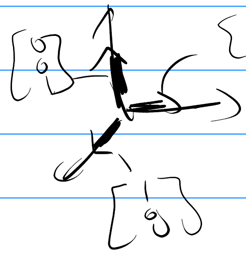
① is the set of all $\mathbf{v}$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k$$

② Spanning set

If $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = V$ (entire vector space)

ex



$$\text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \mathbb{R}^3$$

you call $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ a spanning set of V

3.2 #13

$$\mathbf{x}_1 = \begin{bmatrix}-1\\2\\3\end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix}3\\4\\2\end{bmatrix}$$

$\text{Span}(\mathbf{x}_1, \mathbf{x}_2)$ is all $\mathbf{v} = \alpha_1 \boxed{\begin{bmatrix}-1\\2\\3\end{bmatrix}} + \alpha_2 \boxed{\begin{bmatrix}3\\4\\2\end{bmatrix}}$

If $\mathbf{v} \in \text{Span}$ says α_1, α_2 exist

ex

$$2\begin{bmatrix}-1\\2\\3\end{bmatrix} + 3\begin{bmatrix}3\\4\\2\end{bmatrix} = \begin{bmatrix}7\\16\\12\end{bmatrix}$$

So $\begin{bmatrix}7\\16\\12\end{bmatrix} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$

13a) go?

$$\begin{bmatrix}2\\6\\6\end{bmatrix} \overset{?}{=} a\begin{bmatrix}-1\\2\\3\end{bmatrix} + b\begin{bmatrix}3\\4\\2\end{bmatrix}$$ Span can not happen b/c no soln

$$\begin{bmatrix}-1 & 3\\2 & 4\\3 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2\\6\\6\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-1 & 3 & 2\\2 & 4 & 6\\3 & 2 & 6\end{array}\right] \to \left[\begin{array}{cc|c}-1 & 3 & 2\\0 & 10 & 10\\0 & 11 & 12\end{array}\right]$$

$$\to \left[\begin{array}{cc|c}-1 & 3 & 2\\0 & 1 & 1\\0 & 11 & 12\end{array}\right] \to \left[\begin{array}{cc|c}-1 & 3 & 2\\0 & 1 & 1\\0 & 0 & 1\end{array}\right]$$

$-11r_2 + r_3 = r_3$ no solution!

So $\begin{bmatrix}2\\6\\6\end{bmatrix} \notin \text{Span}(x_1\ x_2)$

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check for Spanning Sets $\quad \text{span}(v_1, \ldots v_k) = V$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \boxed{}$ representation of everywhere in V

ex) check does $\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \mathbb{R}^3$

$$\alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

Solve

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

solve!

Linear Independence

Linear dep

$v_3 = v_1 + v_2$

Note: In spans (linear combos)

ex $\text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right)$

ex $2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

$3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + -1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

All tests for dep/indep

based on Solve:

$$\boxed{\begin{array}{c}\alpha_1 \mathbb{V}_1 + \alpha_2 \mathbb{V}_2 + \cdots + \alpha_k \mathbb{V}_k = \mathbb{O} \\ \text{Only trivial soln?} \to \text{independent} \\ \text{have non-trivial solns?} \to \text{dependent}\end{array}}$$

Ex in $P_{(3)}$ how many terms

is Not degree

any place in P_3

$a + bx + cx^2 = p$

Note: $g = \boxed{x - x^4 + x^3} = 0 + 1\cdot x + 0\cdot x^2 + 1\cdot x^3 + (-1)x^4$

is really

$g \in P_5$

Q: are $2, 3-x, x^2-1$ independent in P_3?

Solve: $\alpha_1(2) + \alpha_2(3-x) + \alpha_3(x^2-1) = 0 + 0x + 0x^2$

$(2\alpha_1 + 3\alpha_2 - \alpha_3) + (-\alpha_2)x + (\alpha_3)x^2 = 0 + 0x + 0x^2$

$$\left\{\begin{array}{l} \alpha_3 = 0 \\ -\alpha_2 = 0 \\ 2\alpha_1 + 3\alpha_2 - \alpha_3 = 0 \end{array}\right. \xrightarrow{\text{Solve}} \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{array}$$

Only trivial soln

So $\boxed{2,\ 3-x,\ x^2-1 \text{ are l.i.d}}$

Wronskian used to check l.i.d. cont. functions.

(ex) e^x, $\sin x$, $\cos^2 x$ in $C[-\pi, \pi]$

Use $\alpha_1 \psi_1 + \alpha_2 \psi_2 + \alpha_3 \psi_3 = \mathbb{0}$

So $\boxed{\alpha_1 e^x + \alpha_2 \sin x + \alpha_3 \cos x = 0}$ Solve? Can't!

like

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$$

$$\alpha_1 f_1' + \alpha_2 f_2' + \alpha_3 f_3' = 0$$

$$\alpha_1 f_1'' + \alpha_2 f_2'' + \alpha_3 f_3'' = 0$$

Solve!

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

how about don't solve it?

if $\det\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix} \neq 0$

→ independence !

$$\left|\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right| \quad \left|\begin{matrix} 1 & & & a \\ & 1 & & c \\ & & 1 & b-a \\ 0 & 0 & 0 & 1 \;\; d-a \end{matrix}\right| \quad \begin{aligned} \alpha_1 &= a-b \\ \alpha_2 &= b \\ \alpha_3 &= c \end{aligned} \quad [\;\;]$$

P_n

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \approx \sin x$

$\alpha_4 = d-a$

$d = \pi \quad a = e$