Math 242

Rules: $D_x[c]$, $D_x[x^p]$, $D_x[c\,f(x)]$

$D_x[f \pm g]$, $D_x[f\,g]$, $D_x[f/g]$

$D_x[\sin x]$, $D_x[\cos x]$, $D_x[\tan x]$

$D_x[\cot x]$, $D_x[\sec x]$, $D_x[\csc x]$

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todo?: ① @ home you should write all these rules

② Make lots of examples with solns.

ex: $D_x[f/g] = \dfrac{f'g - fg'}{g^2}$

$$\frac{d}{dx}\left[\frac{3x^2-2}{\sqrt{x}+x^3}\right] = \frac{(3x^2-2)'(\sqrt{x}+x^3) - (3x^2-2)(\sqrt{x}+x^3)'}{(\sqrt{x}+x^3)^2}$$

$$= \frac{6x(\sqrt{x}+x^3) - (3x^2-2)(\frac{1}{2}x^{-1/2}+3x^2)}{(\sqrt{x}+x^3)^2}$$

$=$ algebra (simplify)

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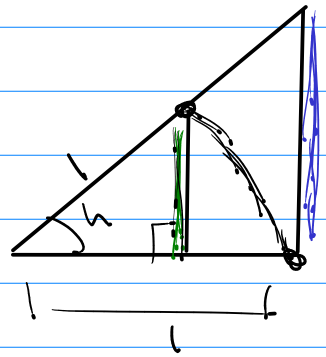
can you show $\lim_{x \to 0} \frac{\sin x}{x} = 1$?



green line < arc < blue line

$$\sin(h) < h < \tan(h)$$

$$\sin(h) < h < \frac{\sin(h)}{\cos(h)}$$

$$1 < \frac{h}{\sin(h)} < \frac{1}{\cos(h)}$$

$$1 > \frac{\sin h}{h} > \cos(h)$$

$\boxed{\text{Composition}}$ $D_x[(f \circ g)(x)] = D_x[f(g(x))]$

ex $\frac{d}{dx}\left[\sqrt{\sin(x)+3}\right] = ?$

⇒ <u>Make a Formula?</u>

$$D_x[f(g(x))]$$

<u>try</u> $= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} = ?$

(stuck)

Ex Study work

$y = f(x)$

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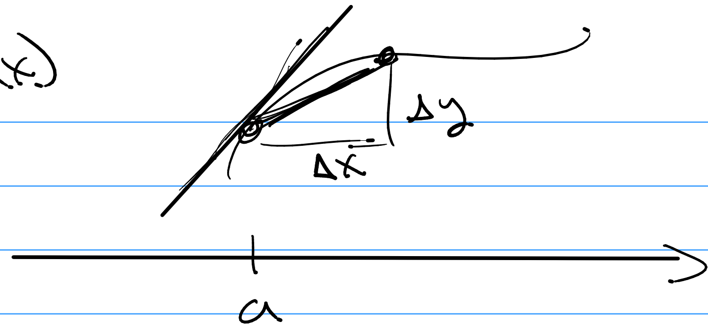


slope of secant ≈ slope of tangent

$$\frac{\Delta y}{\Delta x} \approx f'(a)$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

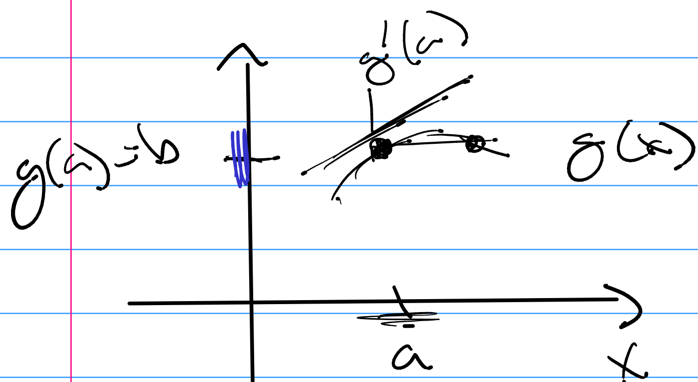
how far secant is from tangent

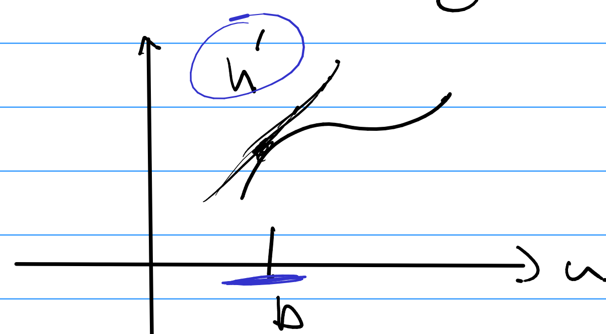
or $$\frac{\Delta y}{\Delta x} - f'(a) = \varepsilon \text{ (error)}$$

so as $\Delta x \to 0$ $\quad \varepsilon \to 0$

Composition ex $h(g(x)) = h(u)$

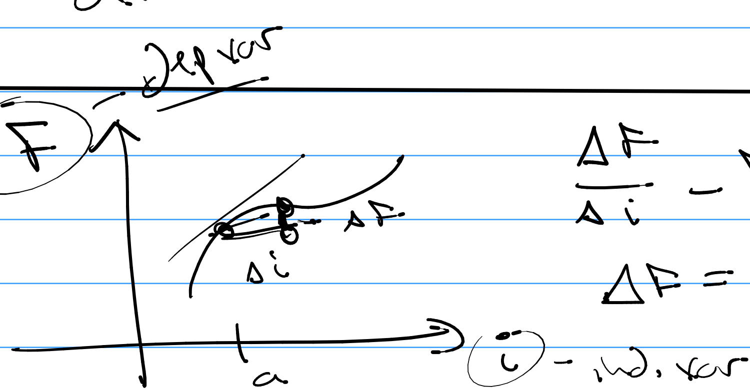
let $u = g(x)$





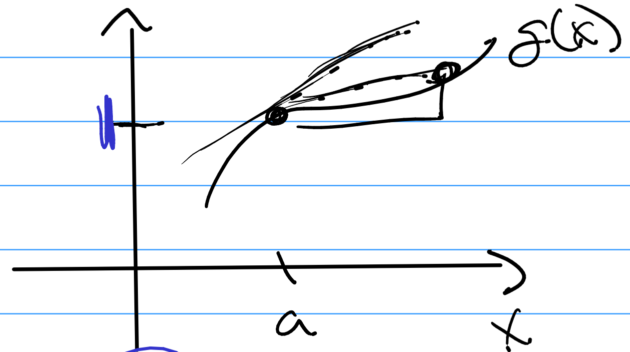
$$\frac{dh}{dx} = ?$$

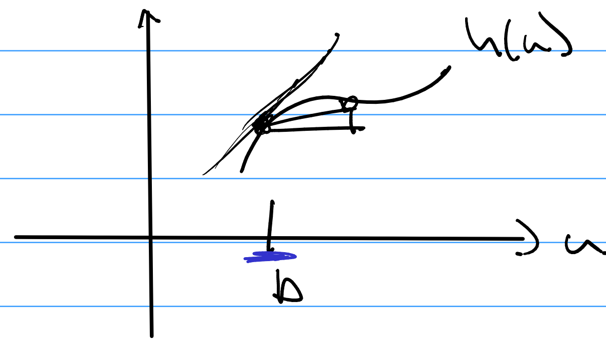
For any function



$$\frac{\Delta F}{\Delta i} - F'(a) = \varepsilon$$

$$\Delta F = (F'(a) + \varepsilon)\Delta i$$





$\Delta g = (g'(a) + \varepsilon_1)\Delta x$ $\quad\quad$ $\Delta h = (h'(b) + \varepsilon_2)\Delta u$

but $u = g(x)$

$\Delta u = \Delta g$

$$\Delta h = (h'(b) + \varepsilon_2)(g'(a) + \varepsilon_1)\Delta x$$

$$\frac{\Delta h}{\Delta x} = (h'(b) + \varepsilon_2)(g'(a) + \varepsilon_1)$$

as $\Delta x \to 0$ $\quad \varepsilon_1 \to 0 \quad \varepsilon_2 \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = h'(b)\, g'(a) \qquad b = g(a)$$

So $D_x[h(g(x))] = h'(g(x)) \cdot g'(x)$

Chain Rule $\quad D_x[f(g(x))] = f'(g(x)) \cdot g'(x)$

esc $D_x\left[\sqrt{x+3x^2}\right] = D_x\left[(x+3x^2)^{1/2}\right]$

$$= \frac{1}{2}(x+3x^2)^{-1/2} \cdot \frac{d}{dx}\left[x+3x^2\right]$$

$$= \frac{1}{2}(x+3x^2)^{-1/2}(1+6x)$$

$$= \boxed{\frac{1+6x}{2\sqrt{x+3x^2}}}$$

$$\frac{d}{dx}\left[\sin\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\right]$$

$\frac{d}{du}\sin(u) = \cos u$

$$= \cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right) \cdot \frac{d}{dx}\left[\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right]$$

$$= \cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right) \cdot \left(-\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right) \cdot \frac{d}{dx}\left[\sqrt{x^2+\frac{x+1}{x-1}}\right]$$

$$= -\cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right) \cdot \frac{1}{2}\left(x^2+\frac{x+1}{x-1}\right)^{-1/2} \cdot \frac{d}{dx}\left[x^2+\boxed{\frac{x+1}{x-1}}\right]$$

$$= \frac{-\cos\left(\cos\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)\right)\sin\left(\sqrt{x^2+\frac{x+1}{x-1}}\right)}{2\sqrt{x^2+\frac{x+1}{x-1}}} \cdot \left(2x + \frac{(1)(x-1)-(x+1)(1)}{(x-1)^2}\right)$$

(ii) $$\frac{d}{dt}\left[\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^4\right]$$

$$= 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3 \frac{d}{dt}\left[\frac{1-\cos(2t)}{1+\cos(2t)}\right]$$

$$= 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3 \cdot \frac{(1-\cos(2t))'(1+\cos(2t)) - (1-\cos(2t))(1+\cos(2t))'}{(1+\cos(2t))^2}$$

Note: $\frac{d}{dt}\left[1-\cos(2t)\right] = \boxed{0 + \sin(2t)\cdot\frac{d}{dt}(2t)}$

$$= 2\sin(2t)$$

$$\frac{d}{dt}\left[1+\cos(2t)\right] = -2\sin(2t)$$

$$\boxed{=} 4\left(\frac{1-\cos(2t)}{1+\cos(2t)}\right)^3 \frac{2\sin(2t)(1+\cos(2t)) + 2(1-\cos(2t))\sin(2t)}{(1+\cos(2t))^2}$$

$$= 8\,\frac{(1-\cos(2t))^3}{(1+\cos(2t))^5}\left(\sin(2t)(1+\cos(2t)) + \sin(2t)(1-\cos(2t))\right)$$

$$= \boxed{16\,\frac{(1-\cos(2t))^3\sin(2t)}{(1+\cos(2t))^5}}$$

Chain rule: $D_x[f(g(x))] = f'(g(x)) \cdot g'(x)$

play with notation: $g(x) = u$

$$D_x[f(u)] = f'(u) \cdot u'$$

ex $\frac{d}{dx}\left[(g(x))^3\right] = 3(g(x))^2 \cdot g'(x)$

let $y = g(x)$

$$\frac{d}{dx}\left[(y)^3\right] = 3(y)^2 \cdot y'$$

$$\frac{d}{dx}\left[y^3\right] = 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

ex $D_x\left[\sin(y) + y \cdot x + x^3\right]$

$y = f(x)$

$$= \cos(y) \cdot y' + \left[(1)y' \cdot x + (y)(1)\right] + 3x^2$$

$$= \cos(y)\, y' + xy' + y + 3x^2$$

Explicit Functions vs Implicit Functions

$y = f(x)$ $\qquad$ $y \neq f(x)$

Ex: $y = x^2 - \sin(x)$ $\qquad$ $\boxed{2y - 3x + 8 = 0}$

$y = \frac{x+1}{x-1}$ $\qquad$ $x^2 + xy - \sin(y^3) = 3x$

$y = \frac{3}{2}x - 4$

$D_x[y] = D_x[\frac{3}{2}x - 4] = \boxed{\frac{3}{2}}$

implicit:

$2y - 3x + 8 = 0$

$\frac{dy}{dx} = ?$

$D_x[2y - 3x + 8] = D_x[0]$

$\frac{dy}{dx} - 3 + 0 = 0$

$\boxed{\frac{dy}{dx} = 3/2}$