# Math 242

**Q5/** $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \sqrt{x+\sqrt{x}}\right)^{-1/2} \boxed{\frac{d}{dx}\left(x + \sqrt{x+\sqrt{x}}\right)}$$

Scratch

$$\frac{d}{dx}\left(x + \sqrt{x+\sqrt{x}}\right) = \boxed{1 + \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2} \frac{d}{dx}\left(x+\sqrt{x}\right)}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \sqrt{x+\sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2} \frac{d}{dx}\left(x+\sqrt{x}\right)\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \sqrt{x+\sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right)$$

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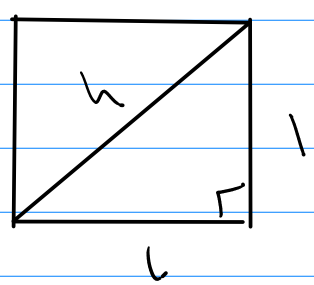
**2.9** ① Linear Approximation (Polynomial Approximation)

② Error




$1^2 + 1^2 = h^2 \to h^2 = 2 \to h \cdot h = 2$

(arithmetic)

$h = \sqrt{2}$

rational number: $\frac{7}{5} = 7 \cdot \left(\frac{1}{5}\right)$

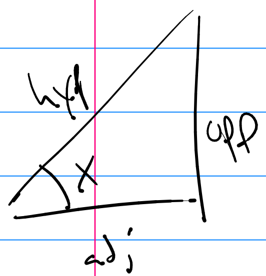
$\sqrt{2} \neq \frac{a}{b}$

$\sqrt{x}$

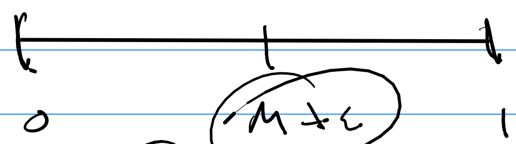
$\sin(x)$, $e^x$, $\ln(x)$ transcendentals

→ transcend algebra.

finite seq of algebra steps.

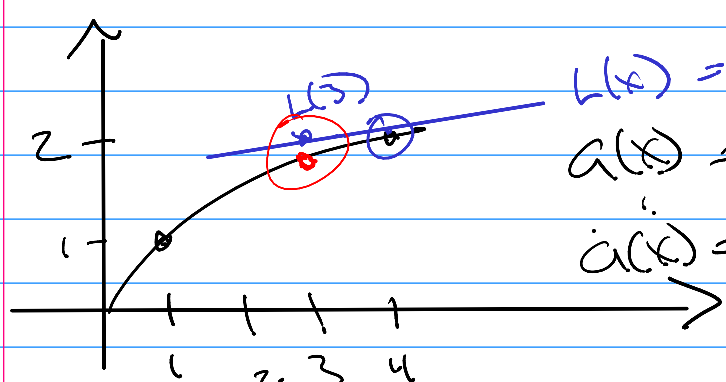



## approximations




$$f(m) = f(m+\varepsilon) \approx ?$$

ex $\sqrt{3} = ?$ → $f(x) = \sqrt{x}$




$L(x) = a + bx$ (Local linear approx)

$Q(x) = a + bx + cx^2$ (local quad. approx)

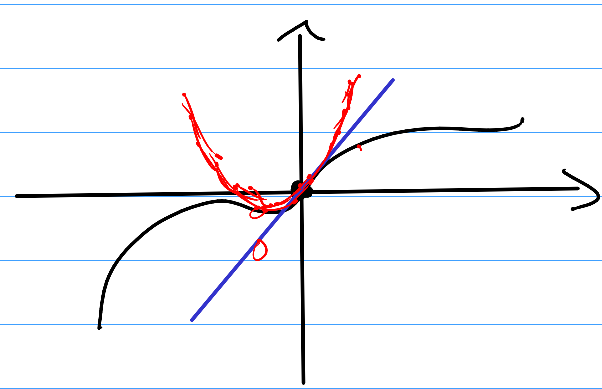
$\hat{Q}(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$ (local poly. of deg n approx)

ex near $x=0$

(1) Match height

(2) (1) + match slope

(3) (1), (2) + match $f^{(2)}(x)$

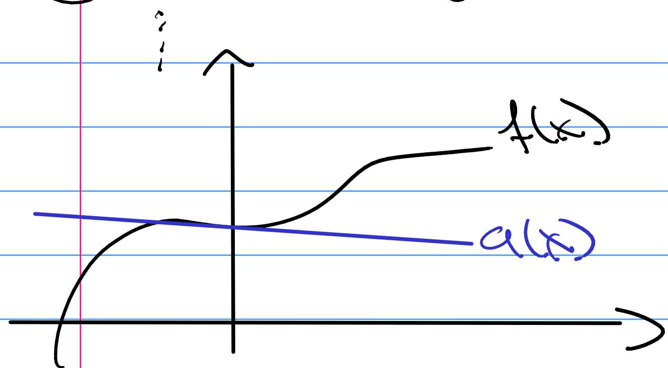


① Same height. $f(0) = a(0)$

② Same $1^{st}$ deriv. $f'(0) = a'(0)$

③ Same $2^{nd}$ deriv. $f''(0) = a''(0)$

?




ex $a(x) = c_0 + c_1 x$

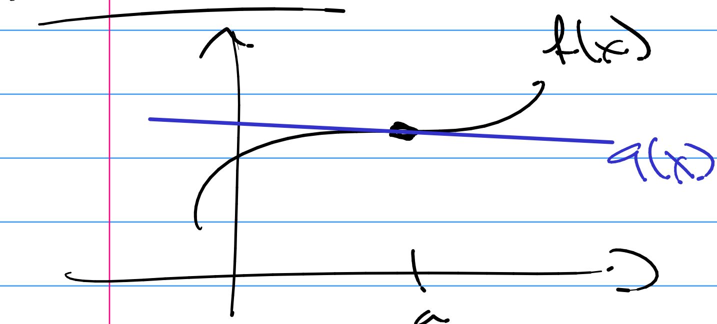
$a'(x) = c_1$

Know: $\boxed{f(0) = a(0) = c_0}$

Know: $\boxed{f'(0) = a'(0) = c_1}$

So $f(x) \approx f(0) + f'(0)\,x$

Move to $x=a$




approx near $x=a$

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

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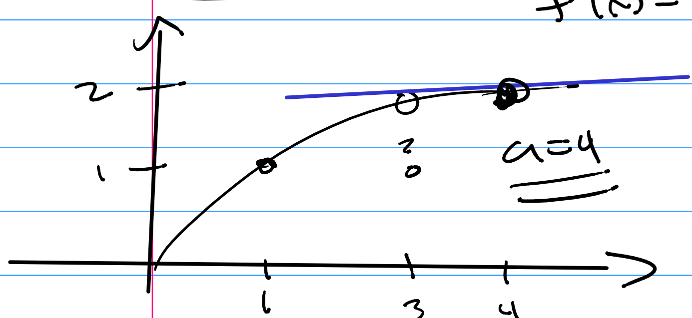
$\sqrt{3} \approx ?$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f(x) \approx f(a) + f'(a)(x-a)$




$a=4$

$$\sqrt{x} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4)$$

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

near $x=4$ $\quad \sqrt{x} \approx 1 + x/4$

$$\sqrt{3} \approx 1 + 3/4 = \frac{7}{4} = \boxed{1.75}$$

check: $\quad \frac{7}{4} \cdot \frac{7}{4} = \frac{49}{16} = 3 + \frac{1}{16} \neq 3$

close!

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$\sin(x)$ near $x=0$ (Linear approx)

$$f(x) \approx f(0) + f'(0)x$$

$f(x) = \sin(x) \to f(0) = \sin(0) = 0$

$f'(x) = \cos(x) \to f'(0) = \cos(0) = 1$

$$\sin(x) \approx 0 + 1 \cdot x = x$$

$$\boxed{\sin(x) \approx x \quad \text{near } x=0}$$

$\sin(0.1) \approx \boxed{0.1}$ $\qquad$ remember: $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$

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Local quadratic: $a(x) = c_0 + c_1 x + c_2 x^2$
(near $x=0$)
$a'(x) = c_1 + 2c_2 x$
$a''(x) = 2c_2$

Know: $a(0) = f(0) = c_0$
$a'(0) = f'(0) = c_1$
$a''(0) = f''(0) = 2c_2 \rightarrow c_2 = \dfrac{f^{(2)}(0)}{2}$

near $x=0$ $\quad f(x) \approx f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2$

near $x=a$ $\quad f(x) \approx f(a) + f'(a)(x-a)\ \dfrac{f''(a)}{2}(x-a)^2$

Polynomial of deg $n$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$
near $x=0$

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$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
near $x=a$

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ex) $3^{rd}$ degree polynomial for $\sin(x)$ near $x=0$

$$f(x) \approx \cancel{f(0)} + f'(0)x + \cancel{\frac{f^{(2)}(0)}{2!}}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f^{(2)}(x) = -\sin x$ $\quad f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(0) = -1$

So $\boxed{\sin(x) \approx x - \frac{1}{6}x^3}$

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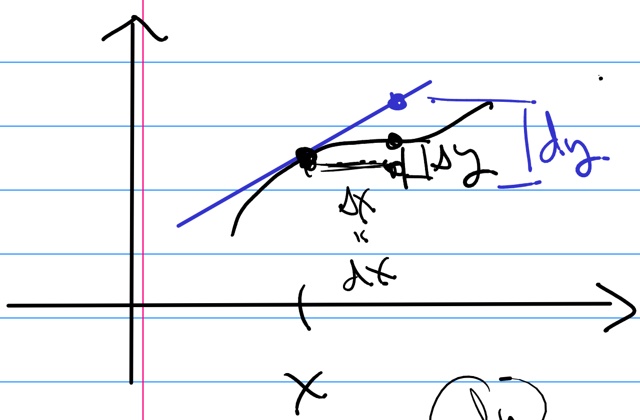
Error Propagation

$x \approx u \pm \text{error}$

$f(x) \approx \boxed{f(u \pm \text{error})}$

real ans $\pm$ error

approx ans $\pm$ error




$\frac{dy}{dx} = f'(x)$

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

differential $\quad dy = f'(x)\,dx$

So $\Delta x = dx$ but $\Delta y = f(x+\Delta x) - f(x)$

$dy = f'(x)\,dx$

and $dy \approx \Delta y$

Error

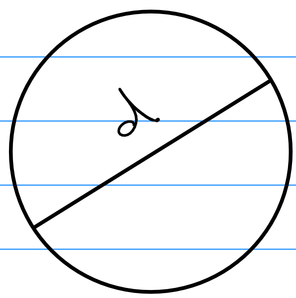
$\Delta x = dx$ is error in measure of ind. variable

$\Delta y$ is true error in outcome of function

$dy = f'(x)dx$ is approx error that you can calculate.

$(\Delta y \approx dy)$

Ex



$C = \pi d$

measure $d = \boxed{3\text{ in}} \pm \boxed{0.1}$ inch

So → $x$ → $dx = \Delta x$

error in Circum.?

$\Delta C \approx dC = (C')(.1)$

$C = \pi d$

$C' = \pi$

$C = \boxed{3\pi} \pm$ error

$C'(3) = \pi$

$C = 3\pi \pm (.1)\pi$

$$\boxed{C = 3\pi \pm 0.314 \text{ (in)}}$$

relative error $\quad \dfrac{\Delta y}{y} \approx \boxed{\dfrac{dy}{y}}$

ex above $C = 3\pi \pm 0.314$

rel. error. $\quad \dfrac{.314}{3\pi} \approx \boxed{0.033}$

$(\%) = \left(\frac{1}{100}\right) \quad \to \boxed{3.3\%}$

to $x \quad \dfrac{.1}{3} \approx \boxed{3.3\%}$