Math 242

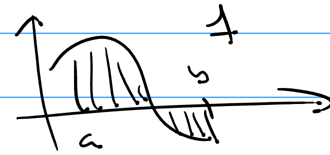
Integral Calculus

① AntiDerivatives (by Knowing Derivative rules..)

Notations a) $A_x[f(x)] = F(x) + C$

b) $\int f(x)\,dx = F(x) + C$ (Indefinite Integral)

② Areas (Net signed areas)



Two ways to find areas..

a) $\int_a^b f(x)\,dx = \lim\limits_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$

b) if you can find $f(x)$'s antiderivative $F(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

4.4 AntiDerivatives / Indefinite Integral notation

again

$$\int f(x)\,dx = F(x) + C$$

because

$$D_x[F(x) + C] = f(x)$$

basis of all our Integration rules

Table

① $\int c f(x)\,dx = c\int f(x)\,dx$

② $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

③ $\int k\,dx = kx + C$

④ $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

⑤ $\int \sin x\,dx = -\cos x + C$

⑥ $\int \cos x\,dx = \sin x + C$

⑦ $\int \sec^2 x\,dx = \tan x + C$

⑧ $\int \csc^2 x\,dx = -\cot x + C$

⑨ $\int \sec x \tan x\,dx = \sec x + C$

⑩ $\int \csc x \cot x\,dx = -\csc x + C$

How to use this table?

① $\int f(x)\,dx = ?$ → a) is $f(x)$ in the table?

(no)

b) it actually is in the table, you need to do algebra / trig / arithm first.

ex) $\int x+3\,dx = \frac{1}{2}x^2 + 3x + C$

ex) $\int x(x+1)\,dx = \int (x^2 + x)\,dx$
$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$

2. $\int_a^b f(x)\,dx = \left[\int f(x)\,dx \right] \Big|_{x=a}^{x=b}$

use table here

ex $\int \frac{1+\sqrt{x}+x}{\sqrt{x}}\,dx = \int \left(\frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}} + \frac{x}{\sqrt{x}} \right) dx$

$= \int \left(x^{-1/2} + 1 + x^{1/2} \right) dx = \boxed{2x^{1/2} + x + \frac{2}{3}x^{3/2} + C}$

check: $\boxed{D_x \left[2x^{1/2} + x + \frac{2}{3}x^{3/2} + C \right] = x^{-1/2} + 1 + x^{1/2}}$

ex $\int_0^{\pi/4} \left(\frac{1+\cos^2\theta}{\cos^2\theta} \right) d\theta = \left[\int \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta \right] \Big|_{\theta=0}^{\theta=\pi/4}$

$= \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \right) d\theta = \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + 1 \right) d\theta$

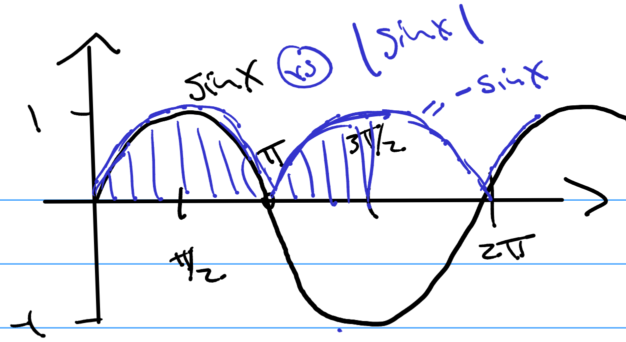
but $\frac{1}{\cos^2\theta} = \sec^2\theta$

$= \int_0^{\pi/4} (\sec^2\theta + 1)\,d\theta = \left[\tan\theta + \theta \right] \Big|_{\theta=0}^{\theta=\pi/4}$

$= \left(\tan(\pi/4) + \pi/4 \right) - \left(\tan(0) + 0 \right)$

$= (1 + \pi/4) - (0 + 0) = \boxed{1 + \pi/4}$

ex $\displaystyle\int_0^{3\pi/2} |\sin x|\, dx$



$$\int_0^{3\pi/2} |\sin x|\,dx = \int_0^{\pi} \underbrace{|\sin x|}_{=\sin x}\,dx + \int_{\pi}^{3\pi/2} \underbrace{|\sin x|}_{=-\sin x}\,dx$$

$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{3\pi/2} \sin x\,dx$$

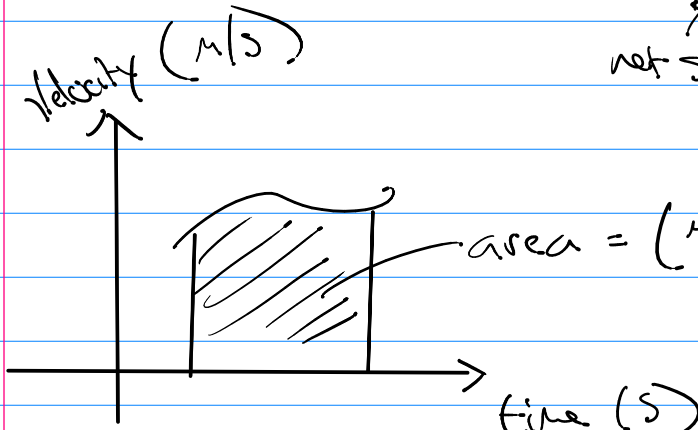
$$= \Big[-\cos x\Big]\Big|_{x=0}^{x=\pi} - \Big[-\cos x\Big]\Big|_{x=\pi}^{x=3\pi/2}$$

$$= \Big[(-\cos(\pi)) - (-\cos(0))\Big] + \Big[(\cos(3\pi/2)) - (\cos(\pi))\Big]$$

$$= [1 + 1] + [0 + 1] = \boxed{3}$$

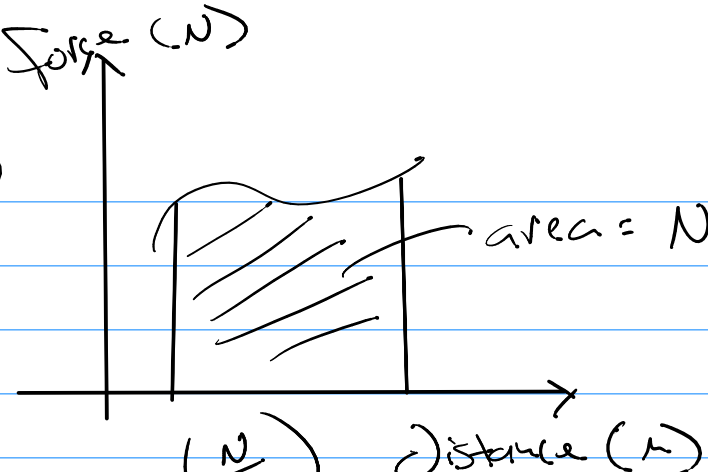
Application Why are areas (net signed area) $\displaystyle\int_a^b f(x)\,dx$ interesting?

ex



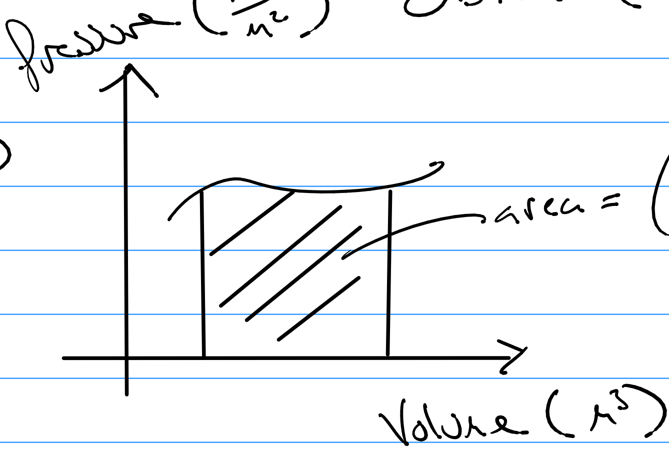
area = (m/s)(s) = meter (Distance)

(2)



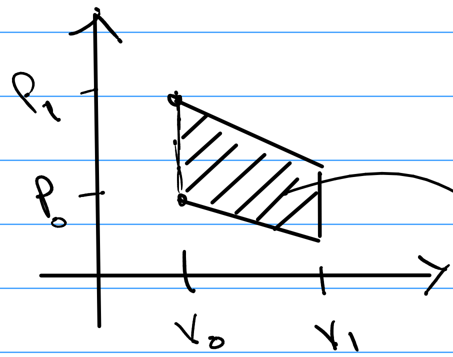
area = N·m = Work

(3)



area = $\left(\frac{N}{m^2} \cdot m^3\right) = N \cdot m$ = Work

PV graph



Stirling Cycle

area = work

Net Change theorem

$$D_x[F(x)] = F'(x)$$

$F(x)$

$$\int_a^b F'(x)\,dx = \underbrace{F(b) - F(a)}_{\text{net change of } F(x)}$$

rate of change of $F(x)$

area under rate of change of $F(x)$

(ex) $V(t)$ is Volume of water in a pool @ time t

$V'(t)$ = rate at which water is draining from pool.

$$\underbrace{\int_{t_1}^{t_2} V'(t)\,dt}_{\substack{\text{area under}\\ \text{the flow } (V') \text{ of}\\ \text{water over } [t_1,t_2]}} = \underbrace{V(t_2) - V(t_1)}_{\substack{\text{net change in Volume}\\ \text{of water.}}}$$

$s(t)$: position

$v(t) = s'(t)$: velocity

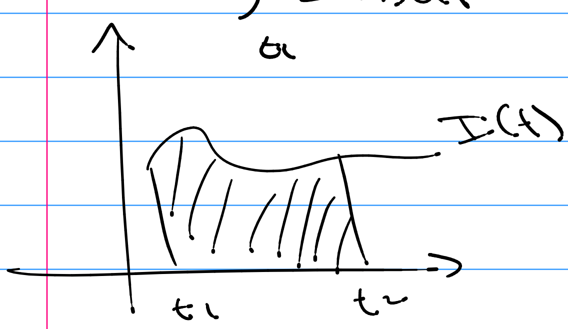
$a(t) = v'(t) = s''(t)$: acceleration

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

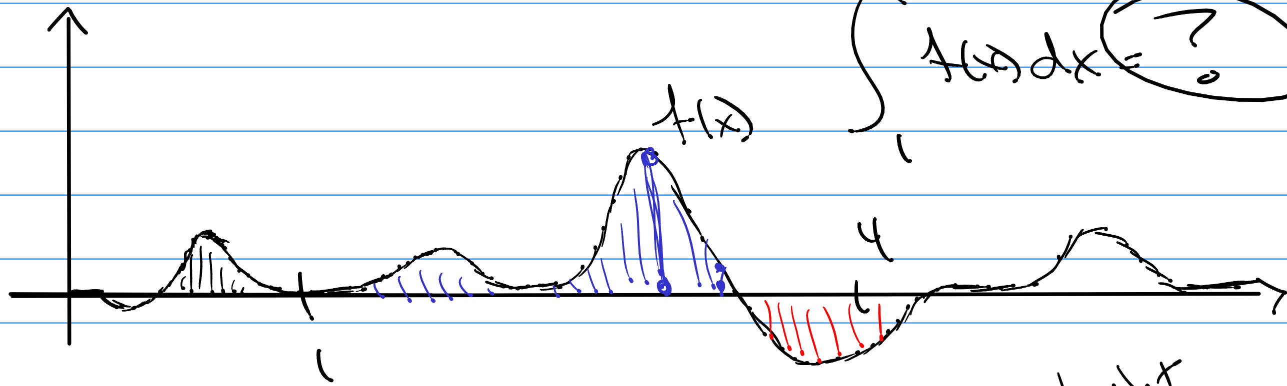
(ex) $D_t[\underset{\text{charge}}{Q(t)}] = I(t)$ ← current

$$\int_{t_1}^{t_2} I(t)\,dt = \underbrace{Q(t_2) - Q(t_1)}_{\substack{\text{net change in charge}\\ \text{from } t_1 \text{ to } t_2}}$$



Ex. $f(x)$ is slope of a walking trail x miles from the start of the trail.

$$\int_1^4 f(x)\,dx = ?$$



$A_x[f(x)] = F(x)$ ← altitude or height

↑ slope $\dfrac{\Delta \text{ height}}{\Delta \text{ distance}} = \dfrac{\text{altitude change}}{\text{walk length}}$

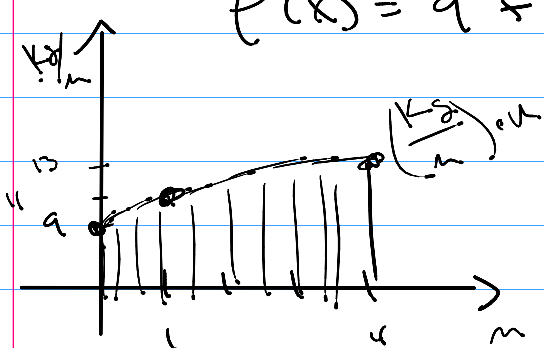
$$\int_1^4 f(x)\,dx = F(4) - F(1)$$

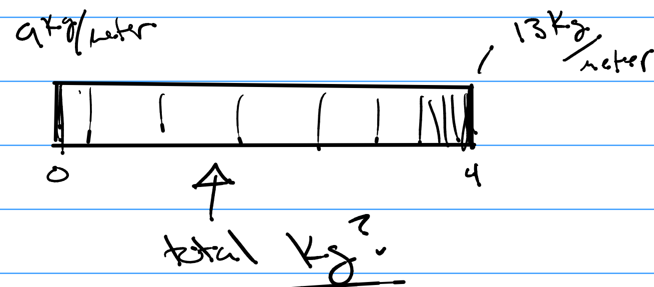
from start

change in altitude or height from mile 1 to mile 4.

#59 linear density of a metal rod 4 m long is

$$\rho(x) = 9 + 2\sqrt{x} \quad \text{kg/meter}$$





$$\int_0^4 (9 + 2x^{1/2})\,dx = 9x + \frac{4}{3}x^{3/2}\Big|_{x=0}^{x=4}$$

$$= \left(9(4) + \frac{4}{3}(4)^{3/2}\right) - (0)$$

$36 + 10 + 2/3$ $\qquad$ $46 + 2/3$ kg