Math 242

$\boxed{3.7}$ Optimization Problems

ex $\quad Y = \dfrac{KN}{1+N^2}$, K is a positive constant

$Y \propto \dfrac{N}{1+N^2}$ $\quad$ ① $N=0 \quad Y=0$

② $\displaystyle \lim_{N\to+\infty} \frac{N}{1+N^2} = \lim_{n\to+\infty} \frac{N}{N^2(1/N^2+1)}$

$\displaystyle = \lim_{n\to+\infty} \frac{1}{N}\cdot\frac{1}{\frac{1}{N^2}+1} = 0\cdot 1 = 0$

Maximize Yield $\quad Y(N) = \dfrac{KN}{1+N^2}$ $\quad$ domain: $(0,+\infty)$

→ look for critical numbers (rel. extrema)

$$Y'(N) = \frac{K(1+N^2) - KN(2N)}{(1+N^2)^2} = K\cdot\frac{1-n^2}{(1+n^2)^2}$$

$Y'(N) = 0$

$1-n^2=0$

$n^2 = 1$

$n = \pm 1$

$\to N = +1$ b/c $\boxed{N \neq -1}$

$Y'(N)$ dne

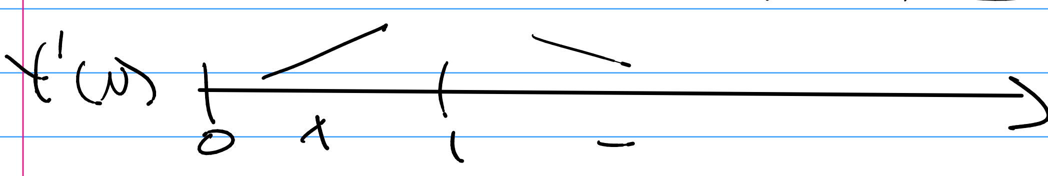
$(1+n^2)^2 = 0$

$1+n^2 = 0$

$n^2 = -1$

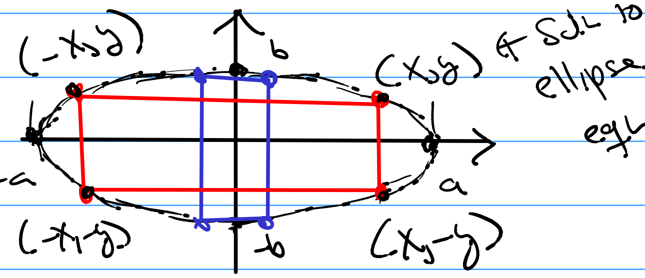
~~$n = \pm i$~~ Never

doesn't apply to $\boxed{\text{this}}$ problem



$$\boxed{\begin{array}{l}\text{rel max @ } N=1 \quad \text{only one rel. extrema} \\ \text{so abs. max @ } N=1 \ \& \ Y(1) = \tfrac{1}{2}K\end{array}}$$

ex $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$



Area of rectangle

$$A = (2x)(2y) = 4xy$$

→ Maximize area means find abs. max of A.

but A has two variables x, y

use $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to substitute for y.

$$y^2 = \left(1 - \frac{x^2}{a^2}\right) b^2 \qquad \sqrt{b^2} = |b|$$

$$y = \sqrt{\left(1 - \frac{x^2}{a^2}\right) b^2} = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Make $A(x) = 4xy = \boxed{\left(\dfrac{4b}{a}x\right)\left(\sqrt{a^2 - x^2}\right) \quad \underline{\text{domain}}: [0,a]}$

→ <u>critical numbers</u> $A'(x) = 0$ or $A'(x)$ dne

$$A'(x) = \frac{4b}{a}\sqrt{a^2 - x^2} + \left(\frac{4b}{a}x\right)\frac{1}{\cancel{2}}\left(a^2 - x^2\right)^{-1/2}(-\cancel{2}x)$$

$$A'(x) = \frac{4b}{a}\sqrt{a^2-x^2} - \left(\frac{4b}{a}x^2\right)\left(\frac{1}{\sqrt{a^2-x^2}}\right)$$

$$= \frac{4b}{a}\left[\frac{\sqrt{a^2-x^2}}{1} - \frac{x^2}{\sqrt{a^2-x^2}}\right]$$

$$= \frac{4b}{a}\left[\frac{a^2-x^2-x^2}{\sqrt{a^2-x^2}}\right] = \frac{4b}{a}\,\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

$A'(x) = 0$

$a^2 - 2x^2 = 0$

$2x^2 = a^2$

$x^2 = \frac{a^2}{2}$

$x = \pm\frac{a}{\sqrt{2}}$ ← not in domain

$\boxed{x = \frac{a}{\sqrt{2}}}$

$A'(x)$ dne

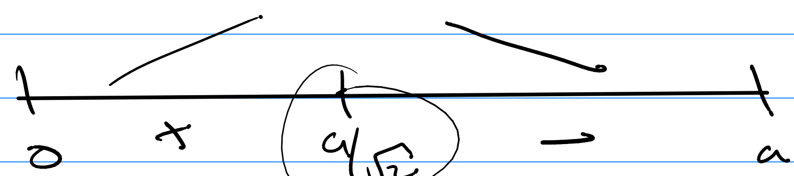
$a^2 - x^2 = 0$

$x = \pm a$ ← not in domain

$\boxed{x = a}$

$$A'(x) = \frac{4b}{a}\,\frac{a^2-2x^2}{\sqrt{a^2-x^2}}$$

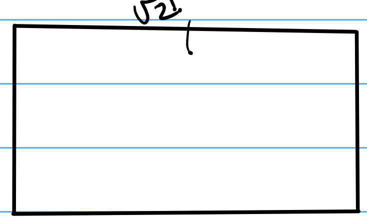
$+$ $\quad$ $+$



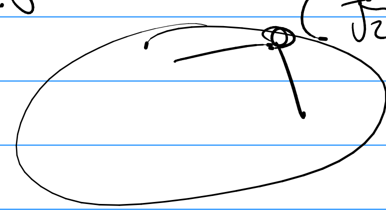
rel max here

→ abs. max @ $x = \frac{1}{\sqrt{2}}a$

$\frac{2}{\sqrt{2}}a = \sqrt{2}a$



$\frac{2}{\sqrt{2}}b = \sqrt{2}b$

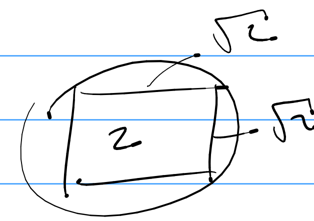


$\left(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}b\right)$

$= x \quad = y$

$\boxed{A = 2ab}$ ← max area

$a = b = 1$



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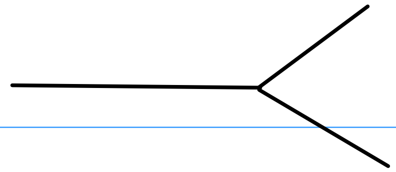
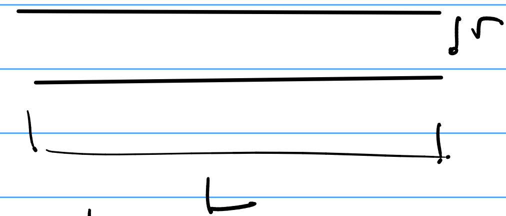


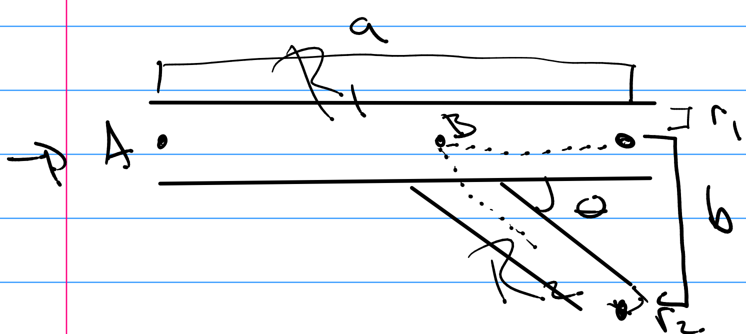
Poiseuille

$R = C\dfrac{L}{r^4}$



$R \propto L \qquad R \propto \dfrac{1}{r^4}$

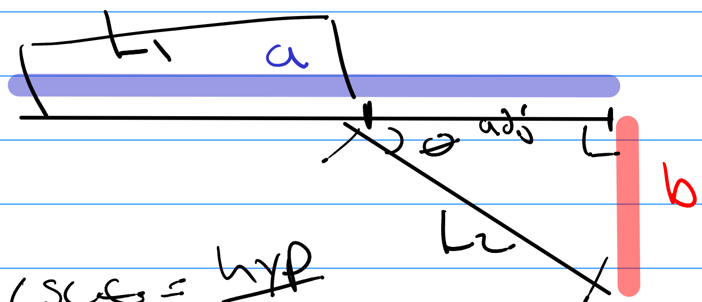
$$\boxed{R = C\frac{L}{r^4}}$$



$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

$$R = C\,\frac{a - b\cot\theta}{r_1^4} + C\,\frac{b\csc\theta}{r_2^4}$$

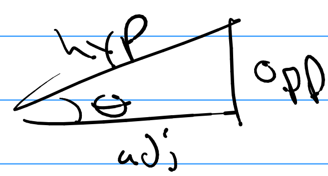
$R_1 \qquad R_2$



$\csc\theta = \dfrac{hyp}{opp}$

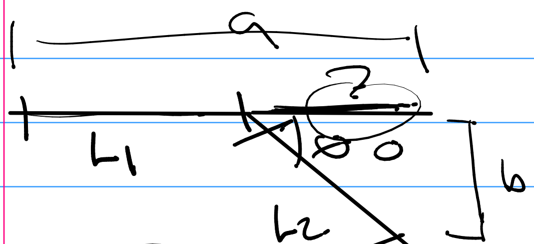
$b\csc\theta = b \cdot \dfrac{L_2}{b} = L_2$

$\sin\theta = \dfrac{opp}{hyp}$



$\cot\theta = \dfrac{adj}{opp}$

$a - b\cot\theta = a - \cancel{b}\dfrac{adj}{\cancel{b}} = a - adj = L_1$



$r_2 < r_1$

$$\boxed{R = C\frac{L_1}{r_1^4} + C\frac{L_2}{r_2^4}}$$

$L_1 = a - b\cot\theta \qquad L_2 = b\csc\theta$

Show min. resistance is when $\cos\theta = \left(\frac{r_2}{r_1}\right)^4$

$$R(\theta) = c\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right) = c\left[\frac{a}{r_1^4} - \frac{b}{r_1^4}\cot\theta + \frac{b}{r_2^4}\csc\theta\right]$$

domain: $\theta \in (0, \pi/2)$

Critical numbers

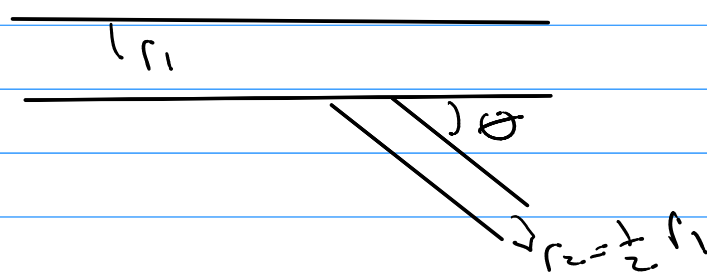
$$R'(\theta) = c\left[0 + \frac{b}{r_1^4}\csc^2\theta - \frac{b}{r_2^4}\csc\theta\cot\theta\right]$$

$$= cb\csc\theta\left[\frac{1}{r_1^4}\csc\theta - \frac{1}{r_2^4}\cot\theta\right]$$

$$R'(\theta) = 0$$

$$\frac{1}{r_1^4}\frac{1}{\sin\theta} - \frac{1}{r_2^4}\frac{\cos\theta}{\sin\theta} = 0$$

$$\boxed{\cos\theta = \frac{r_2^4}{r_1^4}}$$



$$\cos\theta = \left(\frac{\frac{1}{2}r_1}{r_1}\right)^4$$

$$\cos\theta = \frac{1}{16}$$

$$\theta = \cos^{-1}\left(\frac{1}{16}\right)$$

$$\theta = 86^\circ$$

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