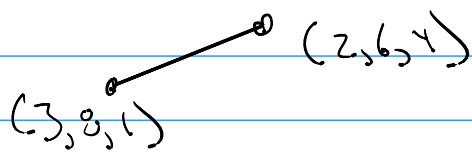
Math 243

Exam 1

(1) $(x-3)^2+(y-8)^2+(z-1)^2=\sqrt{14}$



$r^2=(1)^2+(2)^2+(3)^2$

$= 14$

(2) $x^2+y^2+z^2-2x+4y=11$

$\left[x^2-2x+1\right]+\left[y^2+4y+4\right]+z^2=11+1+4$

$(x-1)^2+(y+2)^2+z^2=16$

(3) $\langle 1,2,3\rangle = a \quad \langle -2,1,2\rangle = b$

a) $|2a-b| = |\langle 2,4,6\rangle - \langle -2,1,2\rangle|$

$= |\langle 4,3,4\rangle| = (4^2+3^2+4^2)^{1/2}$

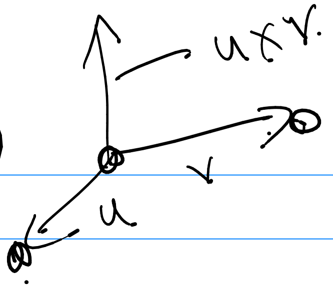
$= \sqrt{41}$

b) $-2+2+6 = \boxed{6}$

(4) $a \perp b \to a\cdot b = 0$ $\quad |a\times b| = |a||b|$

$a \parallel b \to |a\times b| = 0$ $\quad |a\cdot b| = |a||b|$

5.

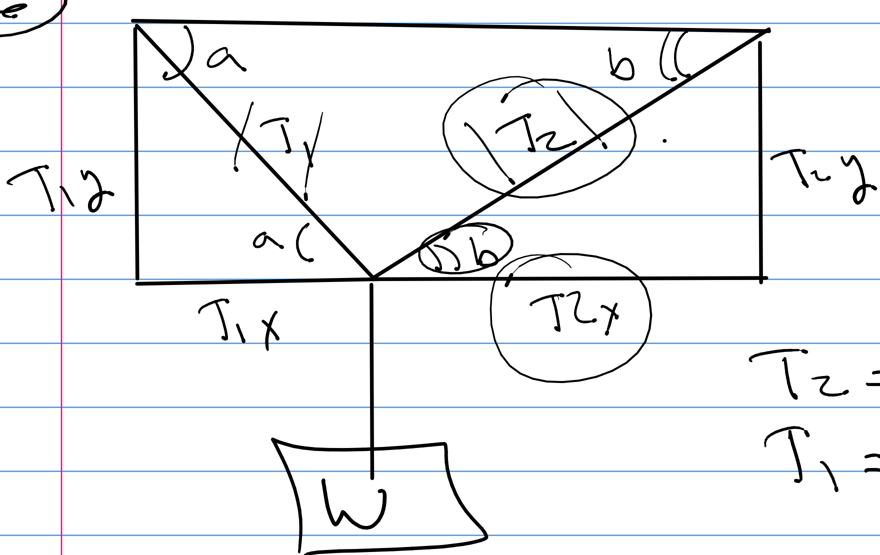


$v = \langle -2, 1, 2 \rangle$

$u = \langle 3, 2, 1 \rangle$

$$v \times u = \begin{vmatrix} i & j & k \\ -2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = \boxed{\langle -3, 8, -7 \rangle}$$

6



$T_2 = \langle T_2x, T_2y \rangle$

$T_1 = \langle -T_1x, T_1y \rangle$

$T_2 = \langle |T_2| \cos b, |T_2| \sin b \rangle$

$T_1 = \langle -|T_1| \cos a, |T_1| \sin a \rangle$

$$\begin{cases} |T_2| \cos b = |T_1| \cos a \rightarrow |T_1| = |T_2| \dfrac{\cos b}{\cos a} \\ |T_2| \sin b + |T_1| \sin a = W \end{cases}$$

$$|T_2| \sin b + |T_2| \cos b \frac{\sin a}{\cos a} = W$$

$$|T_2| = [W / (\sin b + \cos b \tan a)]$$

$$|T_1| = (\quad '' \quad) \frac{\cos b}{\cos a}$$

(7) $W = \vec{F} \cdot \vec{D} = |F||D| \cos\theta$

$= (1200)(1000) \cos(32°)\ N \cdot m$

(8) $|\tau| = |F||r| \sin\theta$

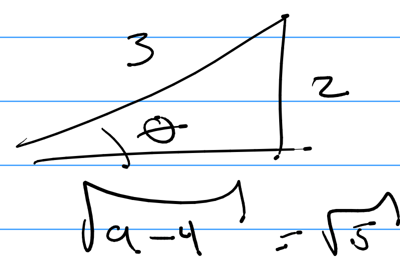
$= (10)(.2) \sin 61°\ N \cdot m$

(9) $a = a_0 e^{kt}$

a) $2 = e^{k(1/4)} \rightarrow k = \boxed{4 \ln 2}$

b) $\boxed{a = 10 e^{4 \ln 2\, t}}$

(10) $\tan(\underbrace{\sin^{-1}(2/3)}_{\theta})$

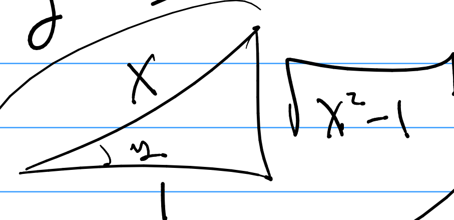


$\parallel$

$\boxed{2/\sqrt{5}}$

(11) $y = \sec^{-1} x \rightarrow \sec y = \frac{x}{1}$

implicit deriv: $\sec y \tan y\ y' = 1$



$y' = \frac{1}{\sec y \tan y}$

$y' = \frac{1}{x\sqrt{x^2-1}}$

$y = \sec^{-1} x \rightarrow \sec y = x \qquad \frac{1}{\cos y} = x$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$

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12 $\ln(x) \sin^{-1}(2x+1) = y$

a) $y' = \left(\frac{1}{x}\right)\left(\sin^{-1}(2x+1)\right) + (\ln x) \frac{1}{\sqrt{1-(2x+1)^2}} \cdot 2$

b) $\lim_{x \to 0} \frac{\tan^{-1}(x)}{\sin(x)} = \lim_{x \to 0} \frac{\frac{1}{1+x^2}}{\cos x} = \boxed{1}$

type $\frac{0}{0}$

13 a) $\int \frac{1}{1+x^2} + \frac{1}{x} + \sin x \, dx$

$= \tan^{-1} x + \ln|x| - \cos x + C$

b) $\int \frac{e^x}{\sqrt{1-(e^x)^2}} \, dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) + C$

let $u = e^x$

$du = e^x dx$

$= \sin^{-1}(e^x) + C$

14. a) $y = \dfrac{\sinh x}{(x^2+3x)}$ $\quad y' = \dfrac{(\cosh x)(x^2+3x) - (\sinh x)(2x+3)}{(x^2+3x)^2}$

b) $\lim\limits_{x\to 0} \dfrac{\tanh x}{\tan x} = \lim\limits_{x\to 0} \dfrac{\operatorname{sech}^2 x}{\sec^2 x} = \lim\limits_{x\to 0} \dfrac{\cos^2 x}{\cosh^2 x}$

type $\frac{0}{0}$ $\quad = \dfrac{1}{1} = 1$

15. $\dfrac{d}{dx}\left[\sinh^{-1} x\right] = \dfrac{1}{\sqrt{1+x^2}}$

$y = \sinh^{-1} x \rightarrow \boxed{\sinh y} = x$

Impl. Deriv. $(\cosh y)\, y' = 1$

$y' = \dfrac{1}{\cosh y}$ b/c $\cosh^2 y - \sinh^2 y = 1$

$y' = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$ ✓

(16) $y = e^{2x} + \ln(x^2) - \sin x + \tanh^{-1}(2x+1)$

$$y' = 2e^{2x} + \frac{2x}{x^2} - \cos x + \frac{1}{1-(2x+1)^2}\cdot 2$$

(17) a) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}(x) + C$

b) $\int \frac{1}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1+(x/2)^2}}\,dx$

let $u = x/2$

$du = \frac{1}{2}\,dx$

$$= \int \frac{1}{\sqrt{1+u^2}}\,du$$

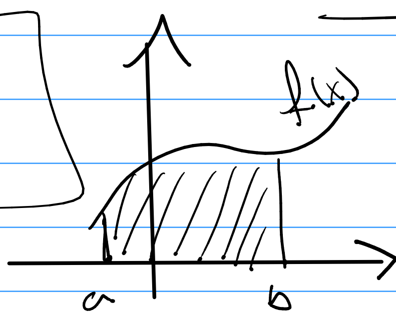
$$= \sinh^{-1}(u) + C = \underline{\sinh^{-1}(x/2) + C}$$

Integration → ch 7 Tech. of Integration.

Indefinite Integral = Antiderivative

Def. Integral = Find the area

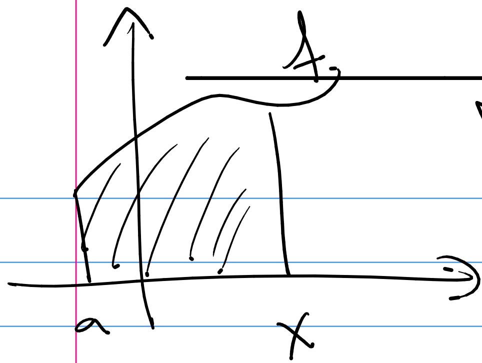
Find, then



Area = ?

$\int f(x)\,dx = \boxed{F(x)+C}$

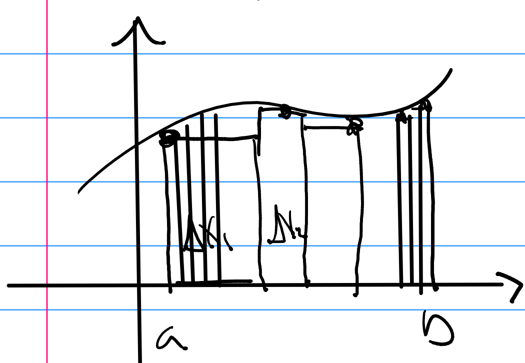
Area = $F(b) - F(a)$



$$F(x) = \int_a^x f(t)\,dt$$

area:

$$\int_a^b f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$



$$D_x[x^2+2] = 2x$$

$$A_x[2x] = x^2+C$$

$$\int (2x)\,dx = x^2+C$$

Antiderivatives

(undo a derivative that has "been done")

(1) Know "all" the derivatives

ex p. 511 " the most important to recognize "

$$\int \frac{1}{x^2+a^2}\,dx = \boxed{\frac{1}{a}\tan^{-1}(x/a)+C}$$

$$\frac{1}{a^2}\int \frac{1}{(x/a)^2+1}\,dx = \frac{1}{a}\int \frac{1}{u^2+1}\,du = \frac{1}{a}\tan^{-1}(u)+C$$

let $u = x/a$ $du = \frac{1}{a}\,dx$

(2) $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x)$

Chain rule $\to$

$$\int \boxed{f'(\boxed{g(x)})\cdot \boxed{g'(x)}}\, dx = f(g(x)) + C$$

let $u = g(x)$ $\qquad \int f'(u)\, du$

$du = g'(x)\, dx$

$= f(u) + C$

$= f(g(x)) + C$

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ex $\int (x \ln x)\, dx = ?$

ex 2 $\int \frac{\ln x}{x}\, dx = \int u\, du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\ln x)^2 + C}$

let $u = \ln x$ $\quad du = \frac{1}{x}\, dx$

ex back to $\int x \ln x\, dx = \int e^{2u}(u)\, du$

$u = \ln x \leadsto e^u = x$

$du = \frac{1}{x}\, dx$

$$\int (x)(\ln x)\,dx$$

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

$$\to \int \left[(f'g) + (fg')\right] dx = fg + C$$

$$\to \int f'g\,dx + \boxed{\int fg'\,dx} = fg + C$$

So $$\boxed{\int fg'\,dx = fg - \int f'g\,dx}$$

$$\int (\ln x)(x)\,dx = (\ln x)\left(\tfrac{1}{2}x^2\right) - \int \left(\tfrac{1}{x}\right)\left(\tfrac{1}{2}x^2\right)dx$$

$$f = \ln x \xrightarrow{\text{deriv}} f' = \frac{1}{x}$$

$$g' = x \xrightarrow[\text{antideriv.}]{} g = \tfrac{1}{2}x^2$$

$$\int (\ln x)(x)\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx$$

$$= \boxed{\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C}$$

Integration by Parts

$$\int \boxed{f(x)\,g'(x)}\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$f(x) \longrightarrow \text{deriv} \rightarrow f'(x)$

$g'(x) \longrightarrow \text{Integrate} \rightarrow g(x)$

$$\int u\,dv = uv - \int v\,du$$

let $u = f(x) \rightarrow \text{deriv} \rightarrow du = f'(x)\,dx$

$dv = g'(x)\,dx \rightarrow \text{integrate} \rightarrow v = g(x)$

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ex $\int (x)(\ln x)\,dx = \frac{1}{2}x^2 \ln x - \int \left(\frac{1}{x}\right)\left(\frac{1}{2}x^2\right) dx$

$f(x) = \boxed{\ln x} \xrightarrow{\text{deriv}} f'(x) = 1/x$

$g'(x) = \boxed{x} \xrightarrow[\text{Integrate}]{} g(x) = \frac{1}{2}x^2$

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ex $\int \ln x\,dx = x \ln x - \int \left(\frac{1}{x}\right)(x)\,dx$

$f(x) = \ln x \xrightarrow{\text{deriv}} f'(x) = \frac{1}{x}$

$g'(x) = 1 \xrightarrow[\text{Integrat}]{} g(x) = x$

$$\int \ln x \, dx = x \ln x - \int dx$$

$$= \boxed{x \ln x - x + C}$$

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2} u^2 + C$$

let $u = \sin x$

$du = \cos x \, dx$

$$= \boxed{\frac{1}{2} \sin^2 x + C}$$
