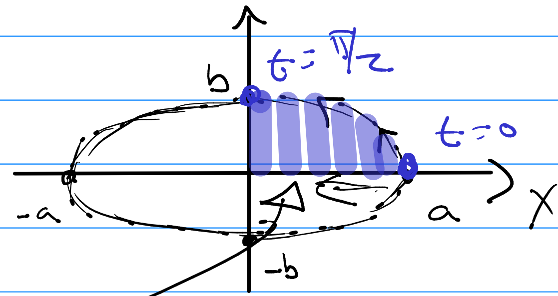
Math 243

Q3/ 10.2 (31)



$x = a\cos\theta \qquad \theta \in [0, 2\pi]$

$y = b\sin\theta$

Area = 4 (area of blue region)

$$= 4\int_{\pi/2}^{0} y\,\underbrace{dx}_{x'\,d\theta} = -4\int_{\pi/2}^{0} ab\sin\theta\sin\theta\,d\theta$$

$$= 4ab\int_{0}^{\pi/2}\sin^2\theta\,d\theta = 4ab\left(\pi/4\right)$$

$$= \boxed{ab\pi}$$

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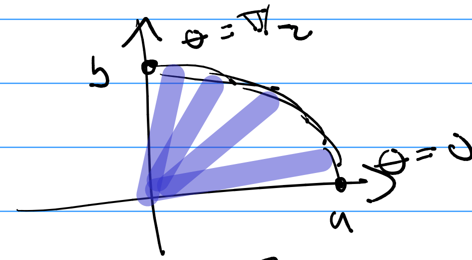
using polar coord.



$x = a\cos\theta$

$y = b\sin\theta$

$$\text{Area} = 4\int_{0}^{\pi/2}\frac{1}{2}\overbrace{r^2}^{=x^2+y^2}\,d\theta = 2\int_{0}^{\pi/2} a^2\boxed{\cos^2\theta} + b^2\boxed{\sin^2\theta}\,d\theta$$

$$= \pi\,\frac{(a^2+b^2)}{2}$$

$\neq ab$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

ex p.704 a) $r = 1 + \sin\theta$ slope of tangent @ $\theta = \pi/3$

b) $\frac{dy}{dx} = 0$ $\frac{dy}{dx}$ dne, (Vertical)

$$\begin{matrix} y(t) \\ x(t) \end{matrix} \longrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\left[\begin{matrix} x = (1+\sin\theta)\cos\theta \\ y = (1+\sin\theta)\sin\theta \end{matrix}\right. \rightarrow \frac{dy}{dx} = \frac{y'}{x'} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

slope $\frac{dy}{dx} = \frac{\cos\theta + 2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta - \sin\theta}$

a) $\left.\frac{dy}{dx}\right|_{\pi/3} = \frac{\cos(\pi/3) + 2\cos(\pi/3)\sin(\pi/3)}{\cos^2(\pi/3) - \sin^2(\pi/3) - \sin(\pi/3)} = ?$

b) $\frac{dy}{dx} = 0 \rightarrow (\cos\theta)(1 + 2\sin\theta) = 0$

$\underline{\cos\theta = 0}$ $\underline{1 + 2\sin\theta} = 0$

$\frac{dy}{dx}$ dne $\rightarrow \underbrace{\cos^2\theta - \sin^2\theta}_{1} - \sin\theta = 0$

$1 - 2\sin^2\theta - \sin\theta = 0$

<u>Note</u> $1 - 2x^2 - x = 0$

$2x^2 + x - 1 = 0$

$(2x - 1)(x + 1) = 0$

$\rightarrow$ So $1 - 2\sin^2\theta - \sin\theta = 0$

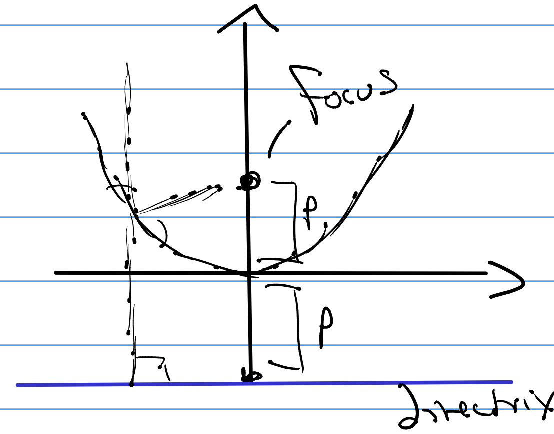
$(\underline{2\sin\theta - 1})(\underline{\sin\theta + 1}) = 0 \rightsquigarrow$ finish

Conic Sections

parabolas:

$$y = \frac{1}{4p} x^2$$

$$x = \frac{1}{4p} y^2$$



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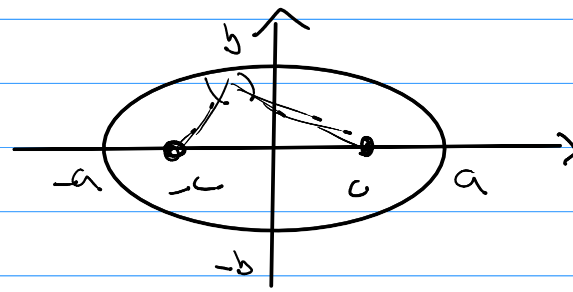
ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 - b^2$$



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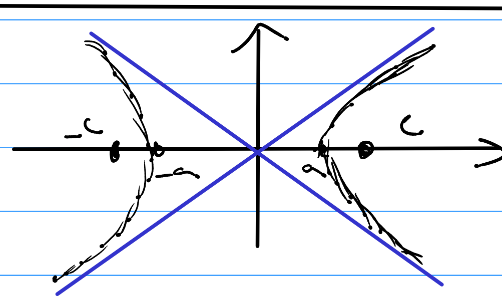
hyperbola

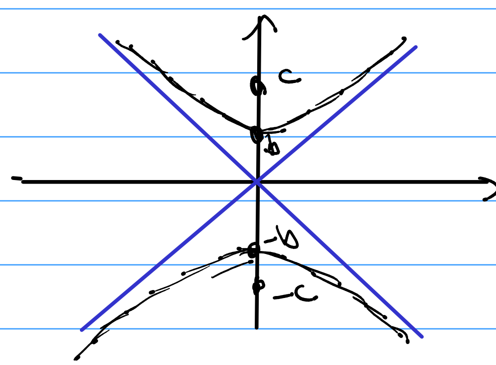
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$$c^2 = a^2 + b^2$$

$$y = \pm\frac{b}{a}x \quad \text{(asym)}$$



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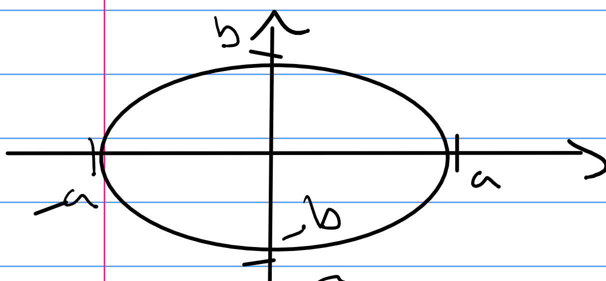
ex) $x^2 - 2x + 2y^2 - 8y + 7 = 0$

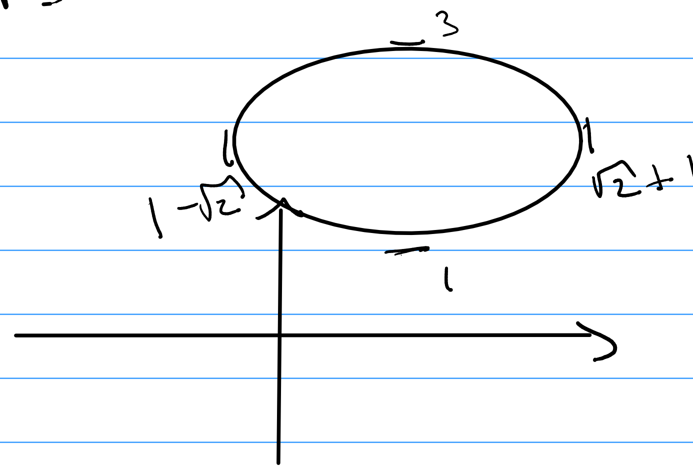
$$(x^2 - 2x + (-1)^2) + 2(y^2 - 4y + (-2)^2) = -7 + 1 + 8$$

$$(x-1)^2 + 2(y-2)^2 = 2$$

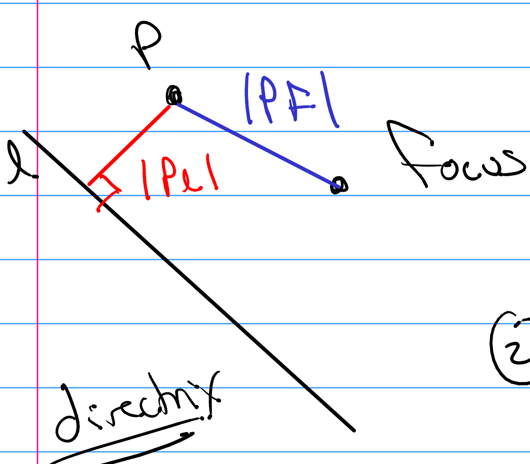
$$\frac{(x-1)^2}{(\sqrt{2})^2} + \frac{(y-2)^2}{(1)^2} = 1$$ ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



10.6 Polar Coordinates



① parabola $|PF| = |P\ell|$

$$\frac{|PF|}{|P\ell|} = 1$$

② ellipse $|PF| < |P\ell|$

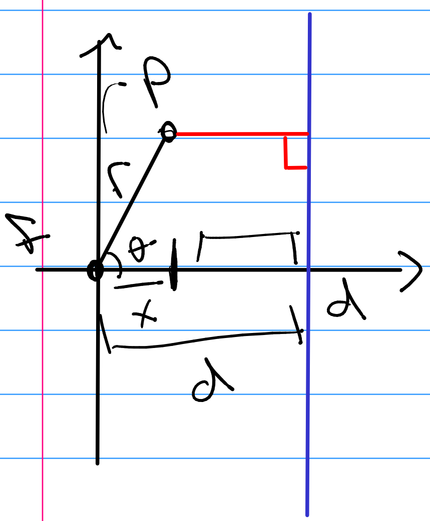
$$\frac{|PF|}{|P\ell|} < 1$$

③ hyperbola $|PF| > |P\ell|$

$$\frac{|PF|}{|P\ell|} > 1$$

Thm let $\boxed{\frac{|PF|}{|Pl|} = e}$ $\quad e \equiv$ eccentricity

1. $e = 1 \rightarrow$ parabola
2. $e < 1 \rightarrow$ ellipse
3. $e > 1 \rightarrow$ hyperbola



P is (x, y) or (r, θ)

$$\frac{|PF|}{|Pl|} = e \rightarrow \frac{r}{d - r\cos\theta} = e$$

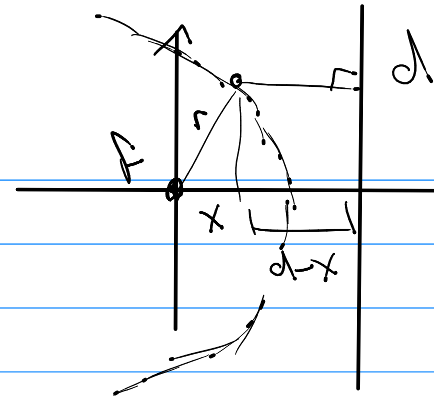
So $r = ed - er\cos\theta$

$$r + er\cos\theta = ed$$

$$\boxed{r = \frac{ed}{1 + e\cos\theta}} \rightarrow$$

$e = 1$ ∴ parabola
$e < 1$ ∴ ellipse
$e > 1$ ∴ hyperbola

$$r = e(d - r\cos\theta)$$



$$r^2 = e^2(d-x)^2$$

$$x^2+y^2 = e^2(d-x)^2$$

$$x^2+y^2 = e^2(d^2-2dx+x^2)$$

$$x^2+y^2 = e^2d^2 - 2e^2dx + e^2x^2$$

$$\boxed{(1-e^2)}x^2 + \boxed{2e^2d}x + y^2 = \boxed{e^2d^2}$$

example: $e=1$ (parabola)

$$2dx + y^2 = d^2$$

$$x = \frac{d^2-y^2}{2d}$$

$$x = -\frac{1}{2d}y^2 + \frac{d}{2}$$ left opening parabola

for $e \neq 1$ (ellipse or hyperbola)

$$(1-e^2)x^2 + 2e^2dx + y^2 = e^2d^2$$

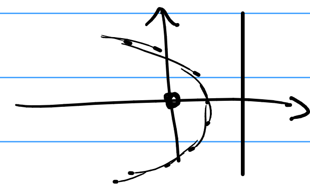
Complete the square

$$(1-e^2)\left[x^2 + \frac{2e^2 d}{1-e^2}x + \left(\frac{e^2 d}{1-e^2}\right)^2\right] + y^2 = e^2 d^2 + \frac{e^4 d^2}{(1-e^2)}$$

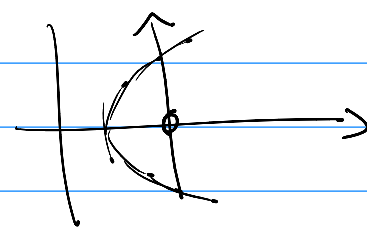
$$(1-e^2)\left(x + \frac{e^2 d}{(1-e^2)}\right)^2 + y^2 = \overset{u}{\boxed{e^2 d^2 + \frac{e^4 d^2}{(1-e^2)}}}$$

$$\frac{\left(x + \frac{e^2 d}{(1-e^2)}\right)^2}{u/1-e^2} + \frac{y^2}{u} = 1$$

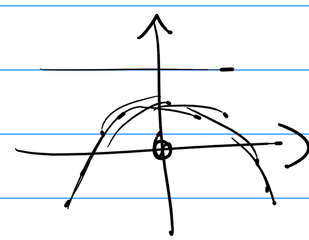
$$r = \frac{ed}{1+e\cos\theta}$$



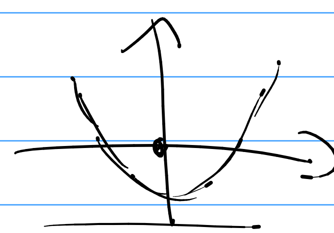
$$r = \frac{ed}{1-e\cos\theta}$$



$$r = \frac{ed}{1+e\sin\theta}$$



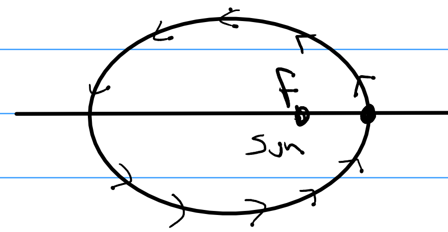
$$r = \frac{ed}{1-e\sin\theta}$$



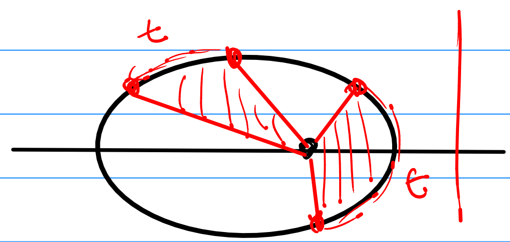
$$r = \frac{ed}{1 \pm e\cos\theta} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

$$\boxed{r = f(\theta)}$$

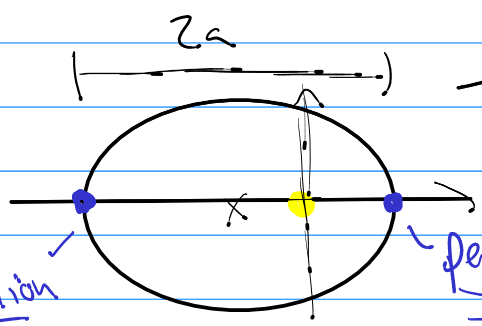
Application Kepler's laws



1. elliptical orbits with sun as on foci
2. equal areas in equal time
3. $(\text{Period})^2 \propto (\text{major axis})^3$



$$r = \frac{ed}{1 + e\cos\theta}$$



$\rightarrow ed = a(1-e^2)$

aphelion $(\theta = \pi)$

$\hookrightarrow a(1+e)$

perihelion $(\theta = 0) \rightarrow a(1-e)$

$$\rightarrow \boxed{r = \frac{a(1-e^2)}{1 + e\cos\theta}} \checkmark$$

Hale-Bopp $\quad e = 0.9951 \quad 2a = 356.5$ AU

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{\frac{356.5}{2}(1-0.9951^2)}{1+0.9951\cos\theta}$$

Max distance from sun? $a(1+e)$

Min distance from sun? $a(1-e)$

Exam 10.1, 10.2, 10.3, 10.4, 10.5, 10.6

90 min **Next Monday**