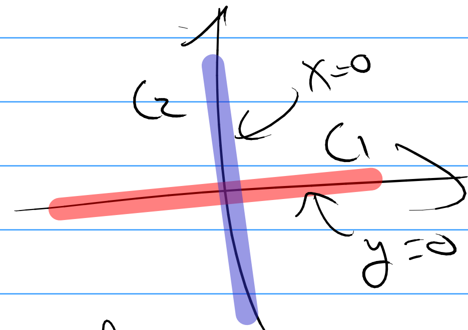
Math 344

11.2

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 - 2y^2}$$



C_1 ($y=0$): $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^4}{x^2} = \lim_{(x,y)\to(0,0)} x^2 = 0$

C_2 ($x=0$): $\displaystyle \lim_{(x,y)\to(0,0)} 2 = 2$

$0 \neq 2$ done.

11.2

① show $\displaystyle \lim_{(x,y)\to(a,b)} f(x,y) = L$

② find $f(a,b)$

③ show $L = f(a,b)$

Continuous.

Assume Cont. $\rightarrow$ Know $\displaystyle \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

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ex $\lim_{(x,y)\to(1,3)} x^3 + \sin(x+y) - e^{xy} = 1^3 + \sin(1+3) - e^{1\cdot 3}$

$= \boxed{1 + \sin(4) - e^3}$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}}$$

Squeeze th$^{\underline{m}}$

$f \le g \le h$ ($f \to L$, $h \to L$)

$\lim g = L$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2} = \overset{?}{0}$$

$$0 \le \frac{x^2 \sin^2 y}{x^2+2y^2} \le \overset{?}{0}$$

b/c f all the squares

Notice: $\dfrac{x^2}{x^2+\underbrace{2y^2}_{+\text{ non-neg}}}$

$x^2 \le x^2 + 2y^2$

$\frac{x^2}{x^2+2y^2} \le 1$

So $$0 \le \underbrace{\frac{x^2}{x^2+2y^2}}_{\le 1} \sin^2 y \le (1)\sin^2 y$$

$$\cancel{0}^{\,0} \le \frac{x^2 \sin^2 y}{x^2+2y^2} \le \cancel{\sin^2 y} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2} = 0}$$

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ex $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = L = 0$$
guess

$$0 \le \left| \frac{xy}{\sqrt{x^2+y^2}} \right|$$

$$\frac{|x||y|}{\sqrt{x^2+y^2}} = \boxed{\frac{|x|}{\sqrt{x^2+y^2}}}|y|$$

$$\frac{|x|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2}}{\sqrt{x^2+y^2}} \le 1$$ Use $\sqrt{x^2} \le \sqrt{x^2+y^2}$

so $\cancel{0}^{0} \le \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \le (1)\cancel{|y|}^{0}$

Prove: $\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0}$ using **$\varepsilon\delta$ proof.**

$0 \le \sqrt{(x-0)^2+(y-0)^2} < \delta \rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \varepsilon$

Show: given $\boxed{0 < \sqrt{x^2+y^2} < \delta}$ ← **you pick**

then $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \varepsilon$

Scratch

$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta$

use $\frac{|x|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2}}{\sqrt{x^2+y^2}} \le 1$

let $\delta = \varepsilon$

Pf given $0 < \sqrt{x^2+y^2} < \delta$

let $\delta = \varepsilon$ so $\sqrt{x^2+y^2} < \varepsilon$

Now: we need to show

goal: $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| < \varepsilon$

$\left| \frac{xy}{\sqrt{x^2+y^2}} \right|$ given: $|x| = \sqrt{x^2}$

and $\sqrt{x^2} \leq \sqrt{x^2+y^2}$

Means: $\frac{|x|}{\sqrt{x^2+y^2}} \leq 1$

$\Rightarrow \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq 1 \cdot |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$

So $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \sqrt{x^2+y^2} \leq \varepsilon$ ∎

given $\delta = \varepsilon$

Partial Derivatives

Ch 10 $\quad \vec{r}(t)$ → limits, cont.

→ derivatives, integrals

Ch 11 $\quad f(x,y)$, $f(x,y,z)$, $f(x_1, x_2, \dots, x_n)$

→ limits, cont.

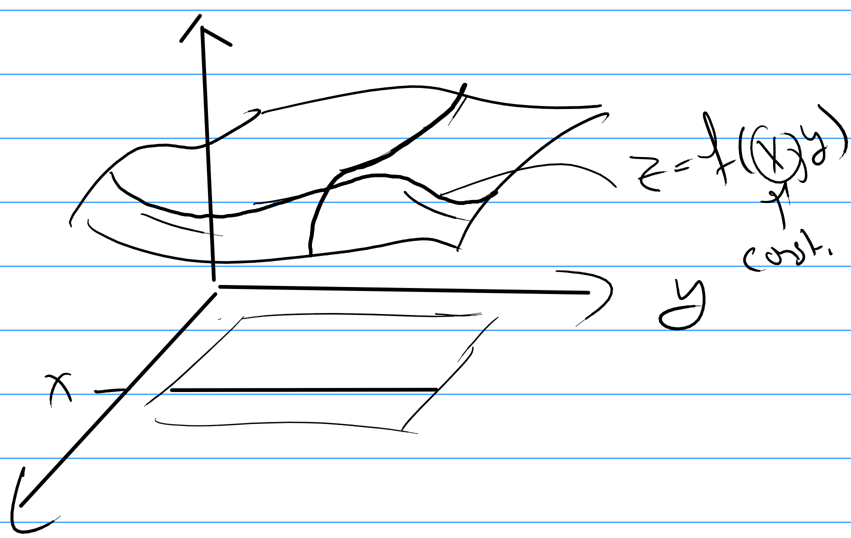
→ **derivatives** (**change**)

Calc 1

$$\text{change in } f = \lim_{\text{scalar} \to 0} \frac{\text{function}(a + \text{scalar}) - \text{fun}(a)}{\text{scalar}}$$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y)$



Partial Derivatives:

$f(x,y)$

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[f] = f_x = D_x f$

Warning Partial vs Implicit

Implicit Derivatives (Apply to equations)

$$x^2 + xy + x^2 \sin y = y$$

y'?

$$2x + [(1)y + xy'] + [2x\sin y + x^2(\cos y)y'] = y'$$

Ex $f(x,y) = xy$

Partial Derivs.

$$\boxed{f_x = y}$$

b/c y is a constant according to f_x