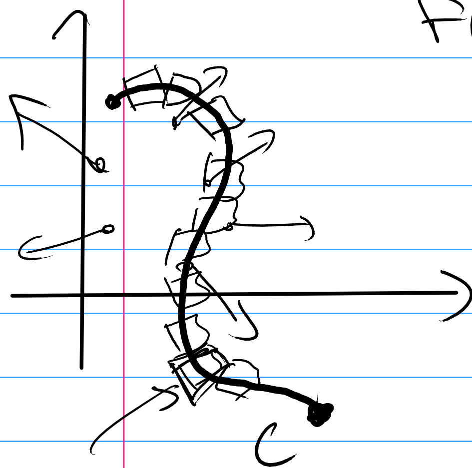
Math 344

$\vec{F}(x,y)$



W_i

Total Work along curve C

Work $=$ $\overrightarrow{\text{Force}} \cdot \overrightarrow{\text{displacement}}$

Scalar — vector field — vector

$W_i = \vec{F}_i \cdot \vec{D}$ $\qquad$ C: $\vec{r}$ traces out C

$\vec{T}\, ds$ ← change in arclength

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$$

$$\text{Work} \approx \sum_{i=1}^{n} W_i = \underbrace{\sum_{i=1}^{n} \left(\vec{F}_i \cdot \vec{T}_i\right) \Delta s}_{\text{Scalar}}$$

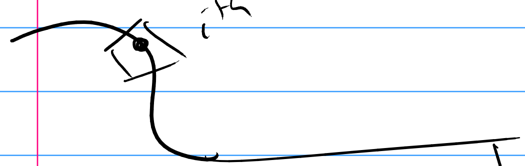
along curve C

$$\Rightarrow W = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} \underbrace{\left(\vec{F}_i \cdot \vec{T}_i\right)}_{\text{Scalar}} \Delta s_i$$

along curve C

① f is a scalar function

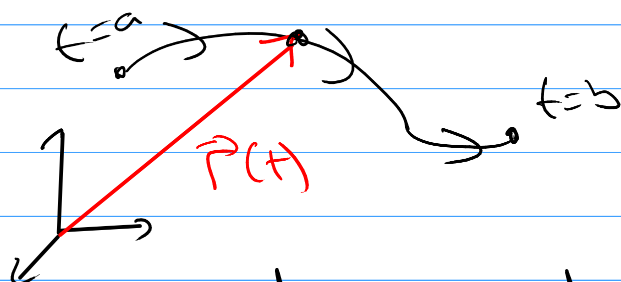
2D $$\int_C f\,ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$



3D $$\boxed{\int_C f\,ds} = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$$

How? $$\int_C f\,ds$$

① param. C : $\vec{r}(t)$ $a \le t \le b$



② $ds \to dt$

2D $$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt$$

$$ds = |\vec{r}\,'|\,dt$$

$$\rightarrow \boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'|\,dt}$$

2D $\quad \vec{r} = \langle x(t), y(t) \rangle$

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\sqrt{(x')^2 + (y')^2}\,dt$$

3D $\quad \vec{r} = \langle x(t), y(t), z(t) \rangle$

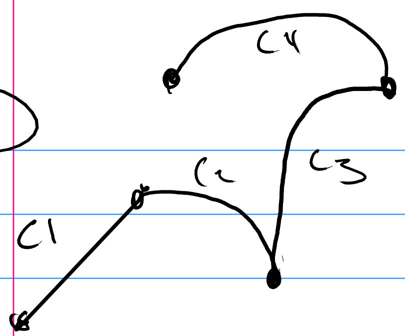
$$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

Tool: line integral of scalar function f along curve C that has parametric eqn $\vec{r}$ from $a \le t \le b$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'|\,dt$$

with respect to arc length

1



$C = C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_n$

finite union of smooth curves.

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$$

2 - Mass of a wire in shape of C

- center of Mass

wire ρ ← density

$$m = \int_C \rho\,ds$$

2D

$$\bar{x} = \frac{1}{m}\int_C x\rho\,ds$$

$$\bar{y} = \frac{1}{m}\int_C y\rho\,ds$$

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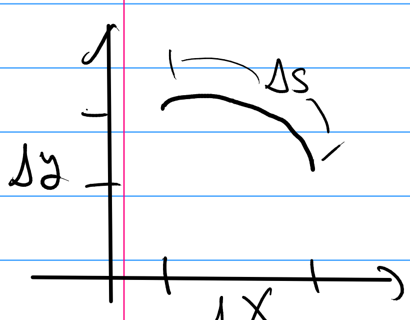
3 w.r.t arc length $\sum f_i \Delta s_i$ ← ds

wrt x $\sum f_i \Delta x_i$

wrt y $\sum f \Delta y_i$



$\boxed{2D}$ $ds = \sqrt{(x')^2 + (y')^2}\, dt$

$dx = \frac{dx}{dt}\, dt = (x')\, dt$

$dy = (y')\, dt$

$$\boxed{\begin{aligned} \rightarrow \int_c f\, dx &= \int_a^b f(\vec{r}(t))(x')\,dt \\ \int_c f\, dy &= \int_a^b f(\vec{r}(t))(y')\,dt \\ \int_c f\, dz &= \int_a^b f(\vec{r}(t))(z')\,dt \\ \int_c f\, ds &= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'|\,dt \end{aligned}}$$

Notation: Normally line integrals w.r.t. x, y, z are together.

$$\int_c (f_1\,dx + f_2\,dy) = \int_c f_1\,dx + \int_c f_2\,dy$$

What about $\vec{F}$?

(start of this class)

$$\text{Work} = \int_C \underbrace{(\vec{F} \cdot \vec{T})}_{} \, ds$$

$\boxed{C: \vec{r}(t)}$ ← scalar function

paramatized C

$$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} \qquad ds = |\vec{r}'| \, dt$$

So $(\vec{F} \cdot \vec{T}) \, ds$

$$= \left(\vec{F} \cdot \left(\frac{\vec{r}'}{\cancel{|\vec{r}'|}}\right)\right) \cancel{|\vec{r}'|} \, dt$$

$$(\vec{F} \cdot \vec{T}) \, ds = (\vec{F} \cdot \vec{r}') \, dt$$

So

$$W = \int_C (\vec{F} \cdot \vec{T}) \, ds = \int_C (\vec{F} \cdot \vec{r}') \, dt$$

$$= \int_a^b (\vec{F}(\vec{r}) \cdot \vec{r}') \, dt$$