Math 322

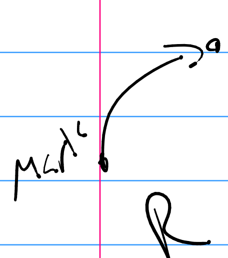
Q's 9.1 7a) $R = \{ (x,y) \mid x \neq y \}$ $\quad$ xRy means $x \neq y$

Ref: $\forall a\,(aRa) \equiv \forall a\,(a \neq a)$ False

Irre $\forall a\,\neg(aRa) \equiv \forall a\,(a = a)$ true

Sym $\forall a \forall b\,(aRb \to bRa)$

$\equiv \forall a \forall b\,(a \neq b \to b \neq a)$ True

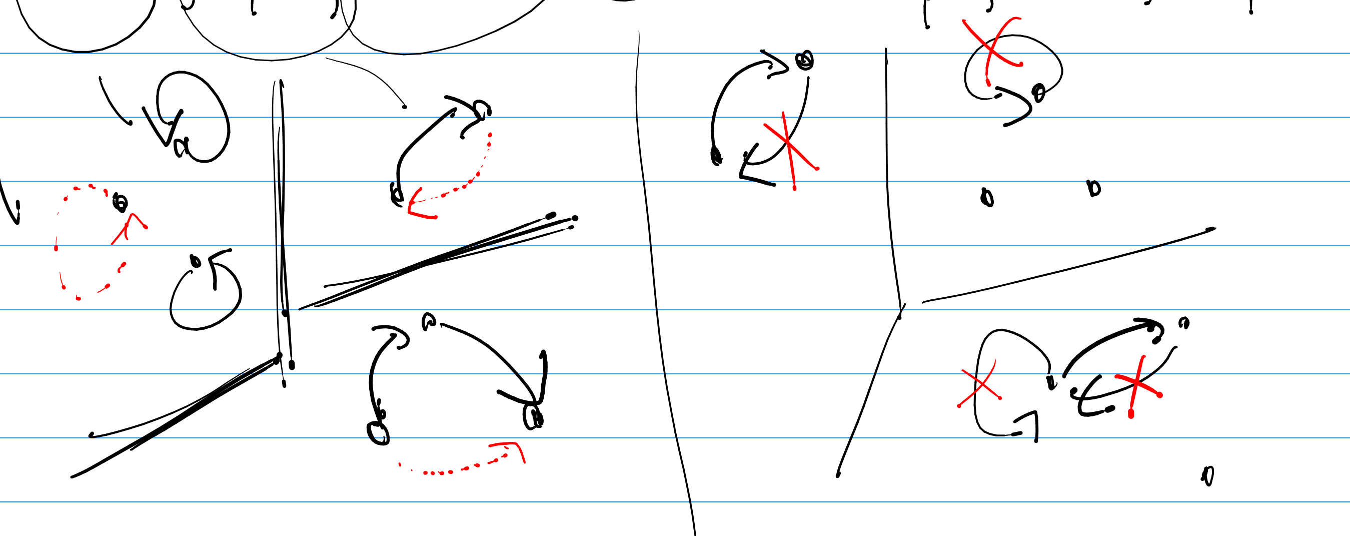


$$\begin{array}{c|cc} & \text{Mark} & \text{Joe} \\ \hline \text{Mark} & 0 & 1 \\ \text{Joe} & 0 & 0 \end{array} = M_R$$

9.4 Know properties ref, irref, sym, etc

ref, sym, trans (6) anti sym, irref, asym

Counter examples



Closure of Relation = New Relation

(old) ∪ (edges)

So the new relation is

reflexive, **symmetric**

or **transitive**

M_R (ex)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

$$\Delta$$

Reflexive Closure: $M_R \vee M_\Delta$

$R \cup \Delta$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Sym Closure: $M_R \vee M_R^T$

$R \cup R^{-1}$

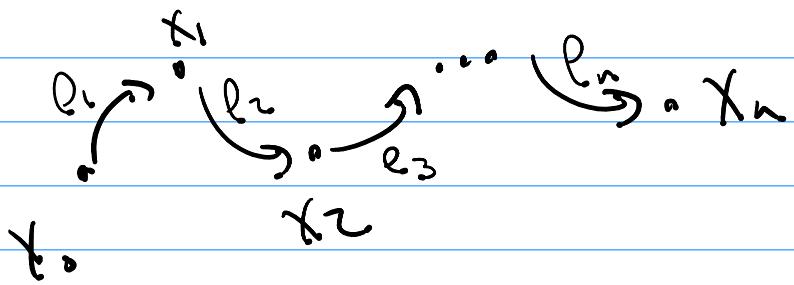
Transitive

R on set A is transitive

iff $\forall n\ R^n \subseteq R'$

Need to understand R^n in a digraph...

(1) Path

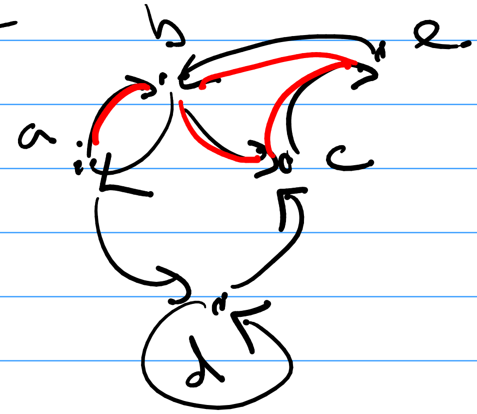


seq of edges $e_1 = (x_0, x_1)$
$e_2 = (x_1, x_2)$
⋮
$e_n = (x_{n-1}, x_n)$

length $= n$

If there is no confusion on which edge you walk...

ex



(a, b)
(b, c)
(c, e)
(e, b)
(b, a)
(a, d)

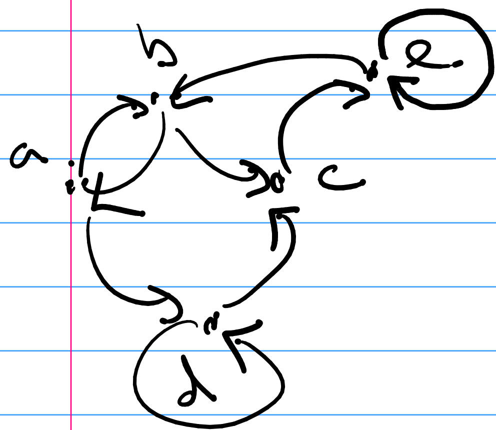
length $= 6$

... use the vertices a, b, c, e, b, a, d

ex a, d, b ← not a path

Terms: length $n \geq 1$ and $x_0 = x_n$

→ **circuit**

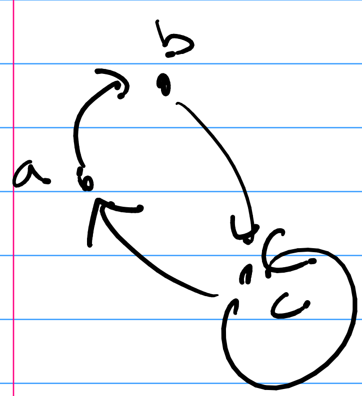


circuit: a, b, a

non-circuit: a, d, c, e

loop: e, e

Thm $(a, b) \in R^n \iff$ there is a path of length n from a to b.



$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_{R^n} = M_R^{[n]}$$

$$M_{R^2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Def R^n the set of all vertices that are connected by path of length n.

$$R^* = R^1 \cup R^2 \cup R^3 \cup R^4 \cup \ldots$$

Def Connectivity relation

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

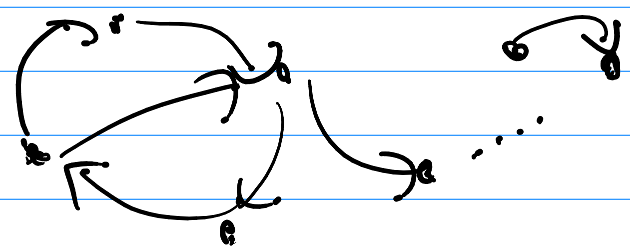
$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \dots$$

Thm R^* is the trans. closure

Problem: R^* need ∞ operations!

Consider R on $A = \{a_1, a_2, \dots, a_n\}$

$|A| = n$



Path: $x_0, x_1, x_2, \dots, x_n$

Thm $|A| = n$ R is a relation on A

if you have a path from a to b

a) $a \neq b$ then there is a path $\leq n-1$

b) $a = b$ then there is a path $\leq n$

So

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}$$

trans. closure