Math 322

$\boxed{\text{Boolean Algebras}}$ (toys (+) rules)

ex

objects: B is a set of two objects $B = \{0, 1\}$

rules: a) two binary operators $\wedge, \vee$

b) one unary operator $\bar{\square}$

Such that the following laws hold.

① Identity laws $\begin{cases} x \vee 0 = x \\ x \wedge 1 = x \end{cases}$

② Complement laws $\begin{cases} x \vee \bar{x} = 1 \\ x \wedge \bar{x} = 0 \end{cases}$

③ Assoc. laws

$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$
$$(x \vee y) \vee z = x \vee (y \vee z)$$

④ comm. laws

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x$$

⑤ distrib laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

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anything that satisfies the above → **Boolean Algebra!**

ex ① Propositional Logic $B = \{F, T\}$

binary ops $x \wedge y$, $x \vee y$
unary op $\neg x$

→ and all 5 laws hold ∴ Boolean Algebra

② "Boolean Algebra" (digital design)

$B = \{0, 1\}$ bits

binary ops $x + y$ bit OR
$x \cdot y$ bit AND
unary $\overline{x}$ bit complement

→ How do they work?

$0 + 0 = 0$ $0 \cdot 0 = 0$
$0 + 1 = 1$ $0 \cdot 1 = 0$
$1 + 0 = 1$ $1 \cdot 0 = 0$
$1 + 1 = 1$ $1 \cdot 1 = 1$

$\overline{0} = 1$ $\overline{1} = 0$

Do the 5 laws hold?

a) Identity $x + 0 = x$
$x \cdot 1 = x$

bit table:

x	0	1	$x+0$	$x \cdot 1$
0	0	1	0	0
1	0	1	1	1

Yes!

e) distrib: $x + (y \cdot z) = (x+y) \cdot (x+z)$

$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

x	y	z	~	$x+(y\cdot z)$	~	$(x+y)\cdot(x+z)$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0				
1	0	1				
1	1	0				
1	1	1				

same

Boolean Algebra of bits $\{0,1\}$ with $x+y$, $x \cdot y$, $\overline{x}$

A) Boolean Expressions

① x is a boolean variable $x \in \{0,1\}$

x
0
1

as truth table

② expression $x + \overline{(y \cdot z)} \cdot 1 + \overline{0}$

any combination of bits, variables, and ops

ex) $(\overline{x+y}) \cdot (\overline{x} + 0)$

x	y	$x+y$	$\overline{x+y}$	$\bar{x}+0$	$(\overline{x+y})\cdot(\bar{x}+0)$
0	0	0	1	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	0	0	0

Notice:

$$\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right. \longrightarrow (\overline{x+y})\cdot(\bar{x}+0) = 0$$

$$(0,1) \xrightarrow{\text{expression}} 0$$
$$(0,0) \longrightarrow 1$$
$$(1,0) \longrightarrow 0$$
$$(1,1) \longrightarrow 0$$

③ Boolean Function:

B^n = n-tuple of 0's and 1's

Boolean Function of degree n

$f(x_1, x_2, \dots, x_n)$ = expression of $x_1, x_2, \dots, x_n$'s

$f: B^n \to B$

Ex $f(x,y) = (\overline{x+y})\cdot(\bar{x}+0)$

by table $f(0,0) = 1$, $f(1,0) = f(0,1) = f(1,1) = 0$

If $f(x_1, x_2, \ldots, x_n)$ is given, you make a table.

x_1	x_2	$\ldots$	x_n	f = expression
0	0	$\cdots$	0	0 or 1 = 2 ways
				0 or 1 = 2 ways
				$\vdots$ $\vdots$
1	1	$\cdots$	1	0 or 1 = 2 ways

2^n rows

$$\boxed{|f| = 2^{2^n}}$$