Math 511

Q's/

$x \perp y$ if $x^T y = 0$

5.2 Orthogonal Subspaces

Def: X, Y are subspaces of V a vector space $\mathbb{R}^n$

Call X orthogonal to Y

if for all $x \in X$ for all $y \in Y$ $x \perp y$ $\rightarrow$ $x^T y = 0$

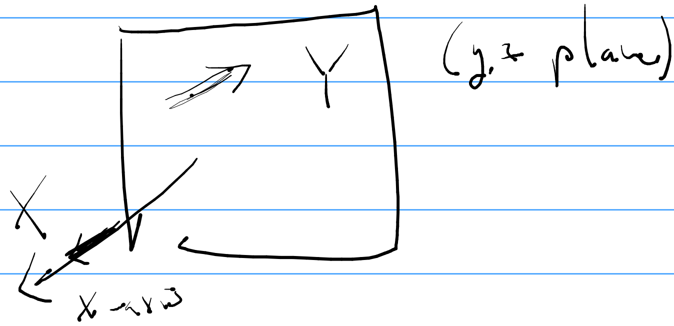
$\rightarrow X \perp Y$

ex $X = \text{span}\begin{pmatrix}1\\0\\0\end{pmatrix} \rightarrow x \in X$ looks like $x = \begin{pmatrix}a\\0\\0\end{pmatrix}$

$Y = \text{span}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) \rightarrow y \in Y$ is $y = \begin{pmatrix}0\\b\\c\end{pmatrix}$

$x^T y = 0 + 0 + 0 = 0$

So $X \perp Y$



Def S is a subspace of $\mathbb{R}^n$

it's orthogonal complement $= S^\perp$

$$S^\perp = \{ \mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n \text{ and } \mathbf{v} \perp (\text{every } \mathbf{s} \in S) \}$$

$$\mathbf{v}^T \mathbf{s} = 0 \text{ for all } \mathbf{s} \in S$$

$\mathbb{R}^5$

$$\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) = X$$

$$\text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right) = Y$$

$$X \perp Y$$

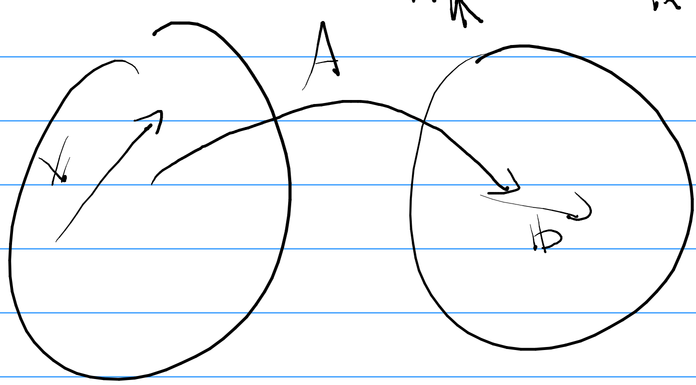
$$X^\perp = Z = \text{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Thm

1. $X \perp Y \rightarrow X \cap Y = \{\mathbf{0}\}$
2. if Y is a subspace $\rightarrow Y^\perp$ is a subspace

A is $m \times n$ consider $A\mathbf{x} = \mathbf{b}$

$\mathbf{x}$ in $\mathbb{R}^n$, $\mathbf{b}$ in $\mathbb{R}^m$



① $A\mathbb{x} = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$ — in col. space of A

So $A\mathbb{x} = \mathbb{b}$ has a soln if $\mathbb{b} \in$ col. space of A

Range of A

$R(A)$

② Consider $A\mathbb{x} = 0$

$\mathbb{x} \in N(A)$

$$N(A) = \{ \mathbb{x} \mid A\mathbb{x} = 0 \}$$

What does $A\mathbb{x} = 0$ look like?

row of A with $\mathbb{x}$

row · col

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = 0$$

$$\begin{aligned} x_1 a_{11} + x_2 a_{12} + \dots + x_n a_{1n} &= 0 \\ x_1 a_{21} + x_2 a_{22} + \dots + x_n a_{2n} &= 0 \\ &\vdots \\ x_1 a_{m1} + x_2 a_{m2} + \dots + x_n a_{mn} &= 0 \end{aligned}$$

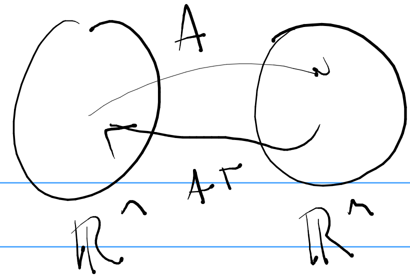
scalar prod

So every row of A is $\perp$ to every $\mathbb{x} \in N(A)$

every col of A^T is $\perp$ to every $\mathbb{x} \in N(A)$

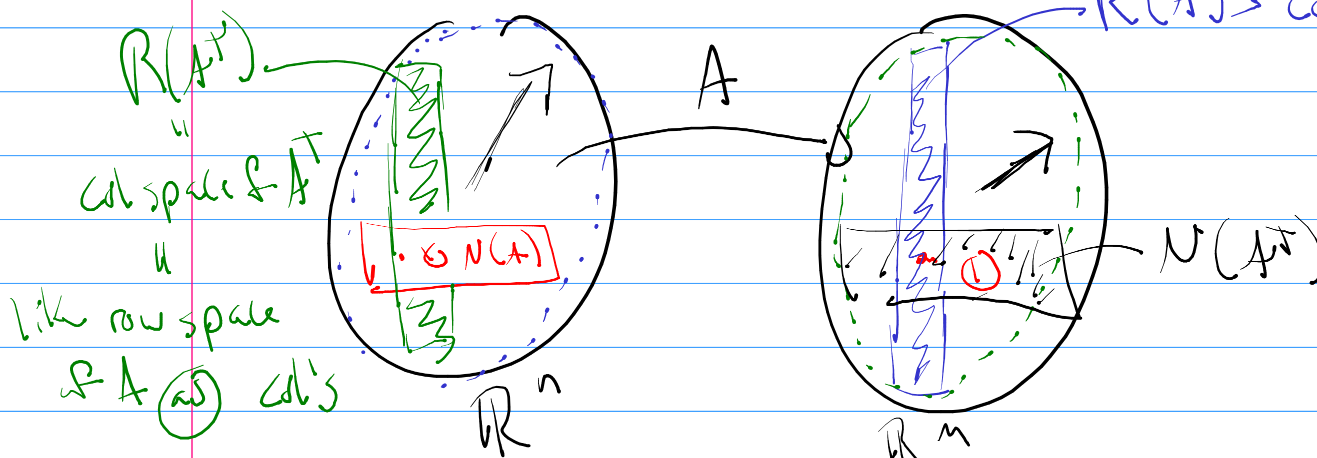
So $\boxed{N(A) \perp \text{col space of } A^T}$

③ Range of A^T = col space of A^T



$$R(A^T) = \{ x \mid x \in \mathbb{R}^n \text{ and } A^T y = x \text{ some } y \in \mathbb{R}^m \}$$

④ So $R(A^T) \perp N(A)$



Th^m Fundamental Subspaces th^m, $A_{m \times n}$

① $N(A) = (R(A^T))^\perp$

② $N(A^T) = (R(A))^\perp$

Th^m S is a subspace of $\mathbb{R}^n$

$\Rightarrow \dim(S) + \dim(S^\perp) = n$

---

if $\mathbb{R}^n$ is broken into S and $S^\perp$

goal: any $v \in \mathbb{R}^n$ is $v = a x + b y$

where $x \in S$ $y \in S^\perp$

a, b are uniq

Def U, V are subspaces of W

and $w \in W$. If $w = au + bv$ $u \in U$ $v \in V$

is uniq $\Rightarrow$ W is a $\boxed{\text{direct sum}}$ of U, V

Notation: $U \oplus V = W$

thm $\mathbb{R}^n = S \oplus S^\perp$ for any subspace S.

thm $(S^\perp)^\perp = S$

So?

① $\mathbb{R}^n = N(A) \oplus R(A^T)$

② $\mathbb{R}^m = N(A^T) \oplus R(A)$

③ $N(A)^\perp = R(A^T)$, $N(A) = R(A^T)^\perp$

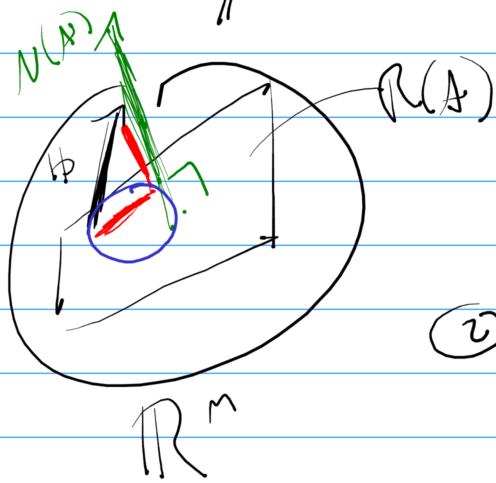
④ $N(A^T)^\perp = R(A)$, $N(A^T) = R(A)^\perp$

$(A\mathbf{x} = \mathbf{b})$

Corollary A is $m \times n$ and $\mathbf{b} \in \mathbb{R}^m$

then either ① $\exists\, \mathbf{x} \in \mathbb{R}^n$ so $A\mathbf{x} = \mathbf{b}$

② $\exists\, \mathbf{y} \in \mathbb{R}^m$ such that...



① $\mathbf{y} \in N(A^T)$

$\rightarrow A^T \mathbf{y} = 0_{\mathbb{R}^n}$

② $\mathbf{y}^T \mathbf{b} \neq 0$

$\rightarrow$ allows a new problem of

$A\mathbf{x} = \mathbf{b}$ has no soln,

New find an $\hat{\mathbf{x}}$ so that $A\hat{\mathbf{x}}$ is close to $\mathbf{b}$

---

also with domain/range restrictions we can invert any matrix, $A_{m \times n}$