Math 511

Q's / $\boxed{1.2\ 6d}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 0 \end{array}\right]$$

free

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array}\right]$$

let $x_4 = a$
(b/c x_4 is free)

$x_1 = -4/3\, a$
$x_2 = 0$
$x_3 = 1/3\, a$

$\boxed{1.4}$

$\boxed{\text{Elm. Alg.}}$

$a \cdot b = 0$ $\qquad$ $(x-2)(x+1) = 0$

$a = 0,\ b = 0$ $\qquad$ $x-2=0 \quad x+1=0$

b/c $\boxed{0 \cdot a = 0}$

vs $3 \cdot a = 1$

$\frac{1}{\cancel{3}} \cdot \cancel{3} \cdot a = \frac{1}{3} \cdot 1$

$1 \cdot a = \frac{1}{3}$

$a = 1/3$

Matrix Algebra $\qquad A \cdot x = b$

if A^{-1} exists $\longrightarrow \left(A^{-1} \cdot A = A \cdot A^{-1} = I \right)$

$$A\mathbf{x} = \mathbf{b} \rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$
$$I\mathbf{x} = A^{-1}\mathbf{b}$$
$$\mathbf{x} = A^{-1}\mathbf{b}$$

$\boxed{1.4}$ all we have is if $M_1 \cdot M_2 = M_2 \cdot M_1 = I$

call M_1, M_2 inverses of each other

1.4 HQ2

$$M_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad M_2 = \frac{1}{a_{11}a_{12} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

are they inv? show $M_1 M_2 = I$

$M_2 M_1 = I$

Consider $A\mathbf{x} = \mathbf{0}$ ← zero vector.

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ ← if this happens we call A singular and A has no inverse

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Def: A^{-1} exists means A is non-singular

A^{-1} doesn't exist means A is singular

1.5 system of eqns $\rightarrow$ solve w/ Aug. matrix (gauss elim) $\rightarrow$ [upper trian | {] back sub to finish

as a Matrix Algebra problem

Solve $A\mathbb{x} = \mathbb{b}$

if magically A^{-1} exists and was known

$$\ldots \quad A^{-1}A\mathbb{x} = A^{-1}\mathbb{b}$$
$$I\mathbb{x} = A^{-1}\mathbb{b}$$
$$\mathbb{x} = \boxed{A^{-1}\mathbb{b}}$$

So if $A\mathbb{x}=\mathbb{b}$ can't be visually solved.

Maybe finding a special M so that

$M A \mathbb{x} = M\mathbb{b}$ could be.

Soln's to $A\mathbb{x}=\mathbb{b}$ vs $MA\mathbb{x}=M\mathbb{b}$

① soln to $A\mathbb{x}=\mathbb{b}$ was $\mathbb{x}_0$

$$\rightarrow \quad MA\mathbb{x}_0 = M\mathbb{b}$$

② soln to $MA\mathbb{x}=M\mathbb{b}$ was $\mathbb{x}_1$

if M^{-1} exists (M is non-singular)

$$M^{-1}MA\mathbb{x}_1 = M^{-1}M\mathbb{b}$$
$$A\mathbb{x}_1 = \mathbb{b}$$

Mult. by non-singular matrices.

Look for 3 non-singular matrices that do the same as our 3 elem. row ops.

(call them the elementary matrices)

① Row Swap E_{type1} = Identity with row_i and row_j swapped

ex $E = \begin{bmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{bmatrix}$ row_2, row_4 swapped $\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix} = I$

$EA = A$ with row_2, row_4 swapped

and $(E_{type1})^{-1} = E_{type1}$

So E_{type1} is non-singular

Note: $A\,E_{type1} = A$ with col_i and col_j swapped.

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② row mult. by m E_{type2}

$E_{type2} = I$ with m in the d_{ii} element

$E_{type2}\,A = A$ with row_i mult. by m.

ex $\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&-3 \end{bmatrix} \begin{bmatrix} 1&2&3\\ 2&1&1\\ -1&0&4 \end{bmatrix} = \begin{bmatrix} 1&2&3\\ 2&1&1\\ 3&0&-12 \end{bmatrix}$

$(E_{type2})^{-1}$ = put $\frac{1}{M}$ in same spot as your orig. E_{type2}

$\boxed{ex}$ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$

so E_{type2} is non-singular

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(3) row$_i$ + M row$_j$ = new row$_i$ E_{type3}

E_{type3} = take I and put M in the row i, col j spot (a_{ij})

$E_{type3}A$ = A with row i = row$_i$ + M row$_j$ (a_{ij} spot)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

row 2
col 1
a_{21}

$(E_{type3})^{-1}$ = put $-M$ in a_{ij} of orig. E_{type3}

$\boxed{ex}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E_{type3} is non-singular

Idea

$A\mathbb{X} = \mathbb{B}$ (allowed to mult. by non-singular matrices)

Note: E_i are E_{type1} or E_{type2} or E_{type3}

$$E_1 A\mathbb{X} = E_1\mathbb{B}$$

$$E_2E_1 A\mathbb{X} = E_2E_1\mathbb{B}$$

$$\vdots$$

$$\underbrace{\boxed{E_n \cdots E_2E_1A}}_{U \text{ upper triangular}}\mathbb{X} = E_n \cdots E_2E_1\mathbb{B}$$

Note: $(E_k \cdots E_2E_1)A = B$

call A, B row equiv. matrices

Th^m A is $n\times n$ then the 3 statements are logically equivalent

1. A is non-singular / invertible
2. $A\mathbb{X} = \mathbb{0}$ has <u>only</u> $\mathbb{X} = \mathbb{0}$ (trivial) soln.
3. A is row equiv. to I

App #1 (find A^{-1})

$E_1 A$ $\qquad\qquad$ $E_1 I$

$E_2E_1 A$ $\qquad\qquad$ $E_2E_1 I$

$\vdots$

$(E_k \cdots E_2E_1)A = I$ $\qquad\qquad$ $E_k \cdots E_2E_1 I$

so $(E_k \cdots E_2E_1) = A^{-1}$

to find A^{-1}

$$[A \mid I] \xrightarrow{\text{row ops}} [I \mid A^{-1}]$$

App #2 factoring.

A

$E_1 A = M_1 \rightarrow A = E_1^{-1} M_1$

$$\underbrace{E_k \cdots E_3 E_2 E_1}\, A = M_k \rightarrow A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1} M_k}_{\text{factoring of } A \text{ into many matrices}}$$

Note: if we restrict E_i to only type 3

$E_k \cdots E_2 E_1 A = U$ (upper triangular)

$$A = \underbrace{(E_1^{-1} E_2^{-1} \cdots E_k^{-1})}_{L \text{ - lower triangular}} U$$

$A = LU$ (LU factorization)