Math 321

Combinations and Permutations → special application of the <u>division</u> rule

→ overcount by product rule → divide to remove it

ex you have n-objects
→ we have $n!$ arrangements of them.

But a) pick r of them to give place awards.

1^{st} place, 2^{nd}, 3^{rd}, ..., r^{th} place

b) we have $(n-r)$ people w/o awards

those people have $(n-r)!$ arrangements that would be an over count of the original $n!$

to pick r for awards with order $= \boxed{\dfrac{n!}{(n-r)!}}$

Pick r from n with order..

Def r-permutation from n objects

$$P(n,r) = \frac{n!}{(n-r)!}$$

(Pick r with order from n)

We could pick r and count it by using the product rule.

$$|\text{pick } r \text{ from } n| = \underset{\text{1st place}}{(n)}\overset{\text{2nd place}}{(n-1)}\underset{\text{3rd place}}{(n-2)}\cdots\underset{r\text{th place}}{(n-r+1)}$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}} = n(n-1)(n-2)\cdots(n-r+1)$$

ex) 5 people, Pick 3 with order

$$\boxed{P(5,3) = \frac{5!}{2!}} = 5\cdot4\cdot3$$

ex) pick 5 to play basketball from 16

$$\boxed{P(16,5) = \frac{16!}{11!}}$$

Combination (choose function)

of n objects choose r without order

$$\text{n objects} \begin{cases} \to \text{choose } r \\ \to \text{do } (n-r) \text{ not choose} \end{cases} \Rightarrow \boxed{\frac{n!}{r!\,(n-r)!}}$$

Def $C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

r-combination from n-objects
or n choose r w/o order

ex choose 10 people to play softball from 16 people = 11 guys + 5 girls.

a) choose 10 (w/o order)

$$C(16,10) = \binom{16}{10} = \boxed{\frac{16!}{10!\,6!}}$$

b) pick 10 (with order)

$$P(16,10) = \boxed{\frac{16!}{6!}}$$

c) choose 10 and exactly 3 must be girls

(How to?) → choose 3 girls and choose 7 guys

$$\binom{5}{3}\cdot\binom{11}{7} = \boxed{\frac{5!}{3!\,2!}\cdot\frac{11!}{7!\,4!}}$$

Adv. Technique

16 people → 5 girls.

Note |all teams| = |no girls| + $\underbrace{\text{|exactly 1 girl| + |exactly 2 girls| + |exactly 3| + |exactly 4| + |exactly 5|}}$

at least 1 girl.

$$|\text{all teams}| = \binom{16}{10} = \frac{16!}{10!\,6!}$$

$$|\text{no girls}| = \binom{11}{10} = \frac{11!}{10!\,1!}$$

ex $$|\text{exactly } 2| = \binom{5}{2}\cdot\binom{11}{8} = \frac{5!}{2!\,3!}\cdot\frac{11!}{8!\,3!}$$

ex $$\boxed{|\text{at least } 3 \text{ girls}|} = |\text{exactly } 3| + |\text{exactly } 4| + |\text{exactly } 5|$$

$$= \binom{5}{3}\cdot\binom{11}{7} + \binom{5}{4}\binom{11}{6} + \binom{5}{5}\cdot\binom{11}{5}$$

$$= \boxed{\text{Finish}}$$

ex at least one girl

$$|\text{all}| - |\text{no girls}| = \boxed{\binom{16}{10} - \binom{11}{10}}$$

= finish using factorial notation

Def: $0! = 1$

why? $\binom{5}{5} = \frac{5!}{5!\,0!} = 1$

one way to send 5 players from 5.

(everyone goes)

$$\frac{\cancel{5!}}{\cancel{5!}\,0!} = 1 \to \frac{1}{0!} = 1$$

So $\boxed{0! = 1}$

Combinatorial Proof or a proof by counting

a task to count

count by tech #1 → ans_1

count by tech #2 → ans_2

So $\boxed{ans_1 = ans_2}$

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back to at least one girl for team of 10 = 11 guys + 5 girls

|all teams| − |all guy teams|

exactly 1 or 2 or 3 or 4 or 5

$$\binom{16}{10} - \binom{11}{10} = \binom{5}{1}\binom{11}{9} + \binom{5}{2}\binom{11}{8} + \binom{5}{3}\binom{11}{7} + \binom{5}{4}\binom{11}{6} + \binom{5}{5}\binom{11}{5}$$

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ex **Pascal's Identity**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

show by algebra.

$$\frac{(n+1)!}{k!(n+1-k)!} \overset{?}{=} \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

by algebra show equality holds.

Show by counting proof.

you have n people plus you → total of (n+1) people

task: choose a committee of k people.

(are **you** in the committee)

yes or no

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$