Math 322

$\boxed{1^{st}}$ HW 13.1(1)
13.2(Finish Example 13.2.3)
13.3(1, 2a, 4b)

$\boxed{Q's}$

Functional Completeness

Boolean function $F(x_1, x_2, x_3, \ldots, x_n) = 0$ or 1

$(n\text{-tuple of } 0,1\text{'s}) \xrightarrow{F} 0 \text{ or } 1$

All of these as a table

$$\begin{array}{cccc|c}
x_1 & x_2 & \cdots & x_n & F \\
\hline
1 & 1 & \cdots & 1 & 0 \text{ or } 1 \\
1 & 1 & \cdots 1 & 0 & \vdots \\
 & \vdots & & & \vdots \\
0 & 0 & \cdots & 0 & 0 \text{ or } 1
\end{array}$$

$\boxed{2^n}$ rows

$$|F| = 2^{2^n}$$

each can be written as minterm or maxterm expansion

So all F can be handled by a finite number of $\wedge, \vee, \bar{\ }$

So set of operators $\{\wedge, \vee, -\}$ is functionally complete.

Note: b/c De Morgan's and Double complement

(a) $xy = \overline{\overline{xy}} = \overline{\bar{x} + \bar{y}}$

so

x	y	F	
1	1	1	$\leftarrow xy$
1	0	0	
0	1	0	
0	0	1	$\leftarrow \bar{x}\bar{y}$

Minterm expansion

$xy + \bar{x}\bar{y}$

Max term expansion

$(\bar{x}+y)(x+\bar{y})$

by functional completeness w/ $xy = \overline{\bar{x} + \bar{y}}$

$$\boxed{xy} + \boxed{\bar{x}\bar{y}} = \boxed{\overline{\bar{x} + \bar{y}}} + \boxed{\overline{\bar{\bar{x}} + \bar{\bar{y}}}}$$

$$= \boxed{\overline{\bar{x} + \bar{y}}} + \boxed{\overline{x + y}}$$

So $\{\vee, -\}$ is functionally complete as well

(b) $x + y = \overline{\overline{x+y}} = \overline{\bar{x}\bar{y}}$

So $x + y = \overline{\bar{x}\bar{y}}$

so $F = (\bar{x}+y)(x+\bar{y}) = ?$

$x + y = \overline{\bar{x}\bar{y}}$

$$(\overline{x}+y)(x+\overline{y}) = (\overline{x\overline{y}})(\overline{\overline{x}y})$$

So $\{\wedge, -\}$ is functionally complete.

$\{\vee, -\}$ is functionally complete

$\rightarrow$ consider a new binary operator NOR

x	y	x NOR y	OR	Not OR
1	1	0	1	0
1	0	0	1	0
0	1	0	1	0
0	0	1	0	1

Not OR $\Rightarrow \overline{x \vee y}$

$$\boxed{\overline{x} = x \text{ NOR } x} = \overline{x}$$

x	x	x NOR x	$\overline{x}$
1	1	0	
0	0	1	

$$(\underline{x \text{ NOR } y}) \text{ NOR } (\underline{x \text{ NOR } y})$$

$$(\overline{x \vee y}) \text{ NOR } (\overline{x \vee y}) = \overline{\overline{x \vee y}} = x \vee y$$

$$\boxed{x \vee y = (x \text{ NOR } y) \text{ NOR } (x \text{ NOR } y)}$$

$$(x \text{ NOR } x) \text{ NOR } (y \text{ NOR } y)$$

$$\overline{\overline{x} \quad \vee \quad \overline{y}} = x \wedge y$$

$$\boxed{x \wedge y = (x \text{ NOR } x) \text{ NOR } (y \text{ NOR } y)}$$

So {NOR} is functionally complete

x	y	F
1	1	1
1	0	1
0	1	0
0	0	0

$(xy) + (x\bar{y})$

$((x \text{ Nor } x) \text{ nor } (y \text{ nor } y)) + ((x \text{ Nor } x) \text{ Nor } (\bar{y} \text{ nor } \bar{y}))$

$= \underbrace{((x \text{ Nor } x) \text{ Nor } (y \text{ nor } y))}_{a} \boxed{+} \underbrace{((x \text{ Nor } x) \text{ nor } ((y \text{ Nor } y) \text{ nor } (y \text{ nor } y)))}_{b}$

$= (a \text{ nor } b) \text{ Nor } (a \text{ nor } b)$

$=$???

NAND = Not and $\qquad \overline{x \wedge y} = x \text{ NAND } y$

$1 \downarrow 0$

x	y	$x \wedge y$	x NAND y
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

$\bar{x} = x \text{ Nand } x$

$x + y = (x \text{ nand } x) \text{ nand } (y \text{ nand } y)$

$xy = (x \text{ nand } y) \text{ nand } (x \text{ nand } y)$

Functionally Complete

$\{\wedge, \vee, \bar{\ }\}$, $\{\wedge, \bar{\ }\}$, $\{\vee, \bar{\ }\}$, {NOR}, {Nand}

$p \to q = \neg p \vee q$

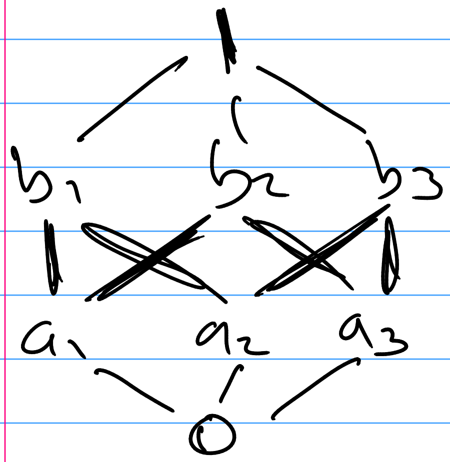
13.4 Finite Boolean Algebras

$[B; \wedge, \vee, -]$ ← come from **partial orders**

↓

which ones?

Lattices that is bounded, distributive, and complemented



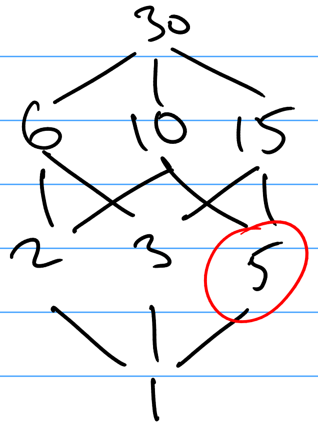
$30 = 2 \cdot 3 \cdot 5$

D_{30} ⟶ factors of 30 $\{1, 2, 3, 5, 6, 10, 15, 30\}$

⟶ makes a Hasse Diagram by | (divisible)

$a \wedge b$

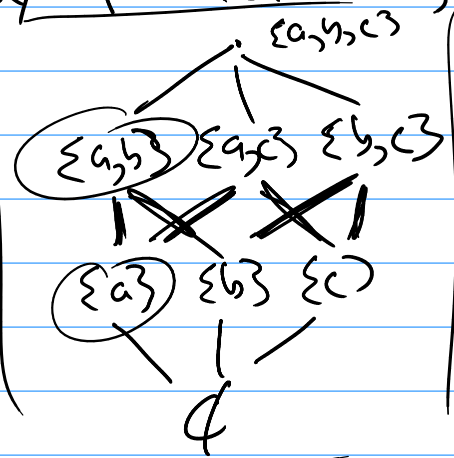
$2 \wedge 6 = 2$



$\bar{5} = 6$

$\{a\} \wedge \{a, b\}$

$= \{a\}$



$\boxed{13.4}$ Every finite Boolean Algebra is isomorphic to an Algebra of Sets.

→ $\{a_1, a_2\}$ $\{a_1, a_3\}$ ...

$A = \{a_1, a_2, a_3, \ldots, a_n\}$

$\{a_1\}$ $\{a_2\}$ $\{a_3\}$ $\{a_n\}$ ← atoms

Consider $P(A)$ of Boolean Algebra

$\emptyset$ ~ empty set is our 0

bottom $\binom{n}{0} +^{\text{row } 1} \binom{n}{1} +^{\text{row}} \binom{n}{2} + \cdots \binom{n}{n} = 2^n$

So

→ Every finite Boolean Algebra has 2^n elements for some n generators (are called atoms)

So Thm: $\{B; \vee, \wedge, {}^{-}\}$ any finite Boolean Algebra let $A = \{a_1, a_2, \ldots, a_n\}$ be the set of atoms of B. Then every non-zero element in B can be expressed uniquely as the join of a subset of A.

So $\boxed{\{P(A); \cap, \cup, {}^{c}\} \text{ is isomorphic}}$ to any $\{B; \wedge, \vee, {}^{-}\}$