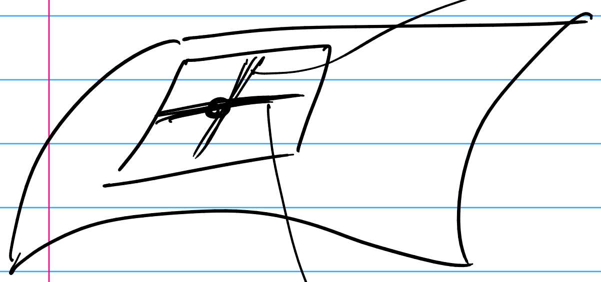
Math 3C14

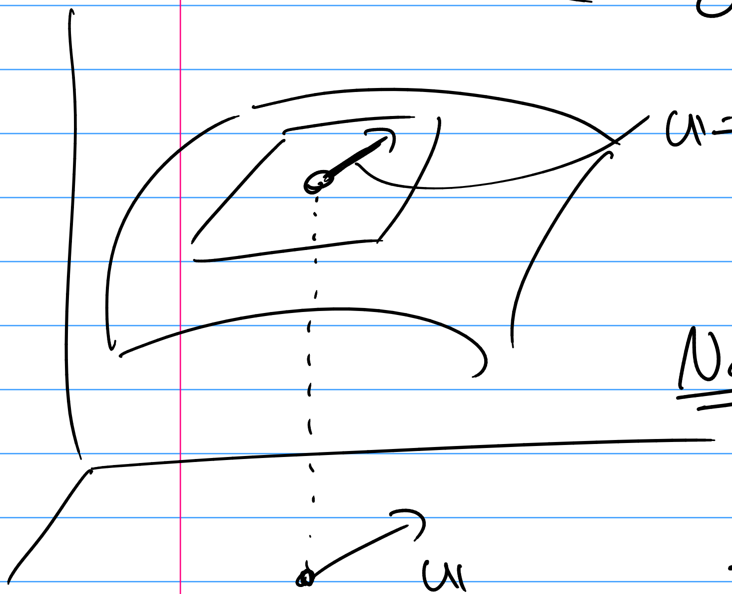
Partial Derivatives

$$\boxed{z = f(x,y)}$$



$$f_x = D_x f = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y = D_y f = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$



$u = \langle a, b \rangle$

Directional Derivative

Notation: $D_u(f)$, u a unit vector

So $f(x,y) + \underline{h \text{ in } u \text{ direction}}$

$\to f(x+ah, y+bh)$

$$D_u(f) = \boxed{\lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}} ??$$

Let $z(h) = f(\underbrace{x+ah}_{x(h)}, \underbrace{y+bh}_{y(h)})$

$$\frac{dz}{dh} = f_x \frac{dx}{dh} + f_y \frac{dy}{dh} = f_x a + f_y b$$

$$z'(h) = f_x(x+ah, y+bh)\, a + f_y(x+ah, y+bh)\, b$$

$$z'(0) = D_u f = \underline{\underline{f_x(x,y)}}\, a + \underline{\underline{f_y(x,y)}}\, b$$

$\underline{\underline{\langle a,b \rangle}} = u$

$$\langle c,d \rangle \circ \langle a,b \rangle = ca + db$$

So $D_u f = \langle f_x, f_y \rangle \circ \langle a,b \rangle$

for u a unit vector $\boxed{D_u f = \langle f_x, f_y \rangle \circ u} = \underline{\underline{\nabla f \circ u}}$

ex $f(x,y) = xy + xy^2$

$f_x = y + y^2$

$f_y = x + 2xy$

$D_u f$ in direction of $\underline{\underline{\langle 1, -2 \rangle}} = u$

$$D_u f = \langle y + y^2, x + 2xy \rangle \circ \langle 1, -2 \rangle / \sqrt{5}$$

$$D_u f = [(y+y^2)(1) + (x + 2xy)(-2)] / \sqrt{5}$$

$$D_u f = \frac{1}{\sqrt{5}}\left(y + y^2 - 2x - 4xy\right)$$

$\langle f_x, f_y \rangle$ is more important than just it's use <u>here</u>.

<u>Def</u>: the gradient of $f(x,y)$ is $\underline{\langle f_x, f_y \rangle}$

<u>Notation</u>: $\nabla f = \langle f_x, f_y \rangle$
"del"

<u>Concept</u> Notation: $\boxed{\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle} f$
∇

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if $f(x,y,z)$ $D_u f = ?$ directional derivative of f in direction u

<u>gives</u> $D_u f = \langle f_x, f_y, f_z \rangle \circ \langle a, b, c \rangle$ & $u = \langle a, b, c \rangle$

$$= \nabla f \circ u$$

so $f(x,y)$ $\quad D_u f = \nabla f \cdot u$ $\quad \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle f$

$f(x,y,z)$ $\quad D_u f = \nabla f \cdot u$ $\quad \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f$

$$f(x,y,z) = \boxed{3x^2y + xyz + z^3}$$

$D_u f$ in direction of $\langle 1,2,3 \rangle$

$$D_u f = \nabla f \cdot u = \langle f_x, f_y, f_z \rangle \cdot \langle 1,2,3 \rangle / \sqrt{14}$$

$$D_u f = \langle 6xy + yz, 3x^2 + xz, xy + 3z^2 \rangle \cdot \langle 1,2,3 \rangle / \sqrt{14}$$

$$= \left((6xy + yz) + (6x^2 + 2xz) + (3xy + 9z^2) \right) / \sqrt{14}$$

$$= \boxed{\frac{1}{\sqrt{14}} \left(9xy + yz + 6x^2 + 2xz + 9z^2 \right)}$$

$z = f(x,y)$

$$D_u f = \langle f_x, f_y \rangle \cdot u$$

in direction of $\langle 1,0 \rangle$ then $D_{\langle 1,0 \rangle} f = \langle f_x, f_y \rangle \cdot \langle 1,0 \rangle = f_x$

so directional derivatives generalize $\underbrace{f_x, f_y}_{\text{special cases}}$ ex

Problem: Maximize Directional Derivative?

$$|D_u f| = |\langle f_x, f_y \rangle \cdot u|$$

perpendicular means dot product $= 0$

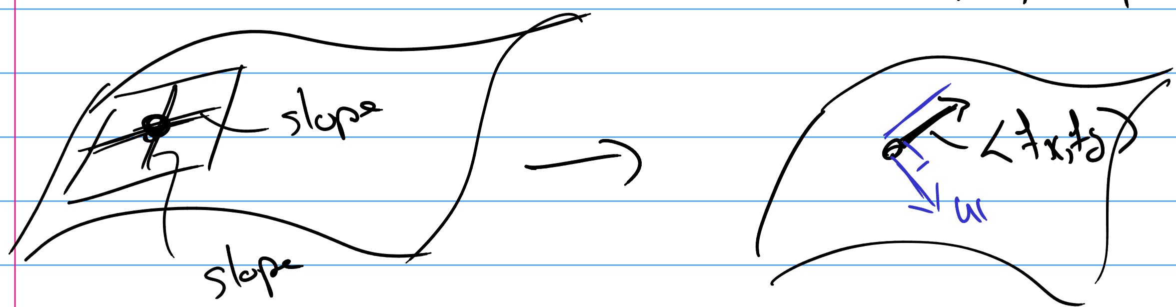
parallel means maximum value of dot prod.

point in same direction

slope slope

Note: @ (x_0, y_0) $\quad \nabla f = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle$

$= \text{constant vector}$



So max of $D_u f$ is $|\nabla f|$

and it occurs when u and ∇f are in the same direction