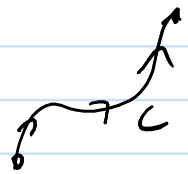
Math 344

$\int_C f\, ds$



C is $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

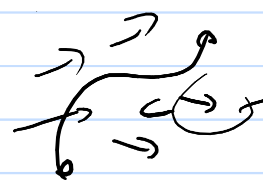
$a \le t \le b$

Know!

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$$

$$= \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$$

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$



$\mathbf{F} = \langle P, Q, R \rangle$ 3D

$\mathbf{F} = \langle P, Q \rangle$ 2D

16.5 Curl, div

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

3D

$\mathbb{F} = \langle P, Q, R \rangle$

ex $\mathbb{F} = \langle xy, x^2 + yz, xyz \rangle$

① $\text{curl}(\mathbb{F}) = \nabla \times \mathbb{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \;,\;,\; \rangle$

$\text{curl}(\mathbb{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2+yz & xyz \end{vmatrix} = \langle xz - y, 0 - yz, 2x - x \rangle$

$= \langle xz - y, -yz, x \rangle$

② $\text{div}(\mathbb{F}) = \nabla \cdot \mathbb{F} = P_x + Q_y + R_z$ $\quad \mathbb{F} = \langle P, Q, R \rangle$

ex $\mathbb{F} = \langle xy, x^2 + yz, xyz \rangle$

$\text{div}(\mathbb{F}) = y + z + xy$

Note: if $\text{div}(\mathbb{F}) = 0$ say incompressible

Properties: $\mathbb{F} = \nabla f$ call $\mathbb{F}$ conservative

and f is its potential function

$\boxed{\text{thm}}$ if $\mathbb{F} = \nabla f$ (so $\mathbb{F} = \langle \underset{P}{f_x}, \underset{Q}{f_y}, \underset{R}{f_z} \rangle$)

and if f's 2^{nd} partials are cont, then

$$\text{curl}(\mathbb{F}) = \nabla \times \mathbb{F} = \nabla \times (\nabla f) = 0$$

$\boxed{\text{thm}}$ $\mathbb{F}$ is defined for all of $\mathbb{R}^3$

and for $\mathbb{F} = \langle P, Q, R \rangle$, P, Q, and R have

cont. 1^{st} partials and $\text{curl}(\mathbb{F}) = 0$

then $\mathbb{F}$ is a conserv. vector field

($\boxed{\text{and}}$ $\mathbb{F} = \nabla f$ so $\begin{cases} f_x = P \\ f_y = Q \\ f_z = R \end{cases}$ Solve by system integration

to get f its potential function)

$\boxed{\text{Th}^m}$ if $\mathbb{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ and P, Q, R have cont. 2nd partials then

$$\text{div}(\text{curl}(\mathbb{F})) = 0$$

$$\nabla \cdot (\nabla \times \mathbb{F}) = 0$$

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before: $\text{curl}(\nabla f) = \nabla \times (\nabla f) = \underline{\underline{0}}$

what about

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f)$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f_x, f_y, f_z \rangle$$

$$= f_{xx} + f_{yy} + f_{zz}$$

Study electrostatics, gravity, fluid dynamics

$$f_{xx} + f_{yy} + f_{zz} = 0$$

f's are called harmonic functions

Notation $\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$

(ex) $\oint_C \mathbb{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

Green's Thm

2D $\quad \mathbb{F} = \langle P, Q \rangle$

Note: $\quad$ 2D to 3D ? $\quad P(x,y)$ $\quad Q(x,y)$

$$\mathbb{F} = \langle P, Q \rangle \longrightarrow \mathbb{F} = \langle P, Q, 0 \rangle$$

Now.. $\operatorname{curl}(\mathbb{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle 0, 0, Q_x - P_y \rangle$

consider: $\quad \langle 0, 0, Q_x - P_y \rangle \cdot \underbrace{\langle 0, 0, 1 \rangle}_{\mathbf{k}} = Q_x - P_y$

So - $\boxed{\oint_C \mathbb{F}\cdot d\mathbf{r} = \iint_D \left(\operatorname{curl}(\mathbb{F}) \cdot \mathbf{k}\right) dA}$

2D $\quad \mathbb{F} = \langle P, Q \rangle \longrightarrow$ 3D $\quad \mathbb{F} = \langle P, Q, 0 \rangle$