Math 322

Due Monday 6.1 (2, 3) 6.2 (2, 3, 6)

Due Wed 6.3 (1, 2, 3, 5, 6, 7, 10)
6.4 (1, 3, 4, 5, 7)
6.5 (1, 2, 3)

(ADS)

Properties: reflexive, irreflexive, sym, antisym, asym, trans.

Equivalence Relation:
R is an equiv. relation iff it is ...
① reflexive
② symmetric
③ transitive.

(ex) $R = \{ (a,b) \mid a \equiv_m b \}$

Def: congruence Notation $a \equiv b \pmod m$
$a \equiv_m b$

① $m \mid (b-a)$
② $a \bmod m = b \bmod m$
③ $a = b + km$, $k \in \mathbb{Z}$

ex Is it an equiv. relation?
a) reflexive? $\forall e\,(eRe)$ "$e \equiv_m e$"

Yes b/c $m \mid (e-e)$
$m \mid 0$

b) <u>Symmetric?</u> $\forall a \forall b \ (aRb \to bRa)$

"$a \underset{m}{\equiv} b \to b \underset{m}{\equiv} a$"

$\cancel{\text{Yes}}$, b/c $m \mid (b-a) \to m \mid (a-b)$

c) <u>trans?</u> $\forall a \forall b \forall c \ (aRb \land bRc \to aRc)$

"$a \underset{m}{\equiv} b \land b \underset{m}{\equiv} c \to a \underset{m}{\equiv} c$"

$m \mid (b-a) \land m \mid (c-b)$

from number theory if $m \mid A \land m \mid B$

$\to m \mid k_1 A + k_2 B$

so $m \mid (b-a) + (c-b) \to m \mid c-a$

so $a \underset{m}{\equiv} c$ $\boxed{\text{Yes}}$

<u>Equivalence Class</u> = all elements equivalent to a given element.

<u>Notation:</u> $[e]_R = \{ s \mid eRs \}$

$\boxed{\text{ex}}$ $[-3] = ?$ if $R = \{(a,b) \mid a \underset{7}{\equiv} b \}$

$[-3] = \{ \ldots, -17, -10, -3, 4, 11, 18, \ldots \}$

<u>Notice.</u> $[-3] = [11] = [-17] = \ldots$

any element in an equivalence class can be its representative.

Also: ① if $a R b$ then $[a] = [b]$

② if any element in $[a]$ is also in $[b]$

$$[a] \cap [b] \neq \emptyset$$

then $[a] = [b]$

So the equivalence class partition our set A

for R on set A.

$$A = [e_1] \cup [e_2] \cup \cdots \cup [e_k]$$

① where you have K equiv. classes

② $[e_i] \cap [e_j] = \emptyset \quad i \neq j$

③ Union of all $[e_i]$ is A.

(ex) $R = \{ (a,b) \mid a \underset{3}{\equiv} b \}$ on $\mathbb{Z}$

$\mathbb{Z}$ $\quad \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

$$\mathbb{Z} = [0] \cup [1] \cup [2]$$

$[0] = \{ \ldots, -6, -3, 0, 3, 6, 9, \ldots \}$

$[1] = \{ \ldots, -5, -2, 1, 4, 7, \ldots \}$

$[2] = \{ \ldots, -4, -1, 2, 5, 8, \ldots \}$

Partial Orderings

If R on a set A is ..

① reflexive
② antisymmetric
③ transitive

we call R a partial order. A is partially ordered by R. Together (A, R) is called a poset.

(ex) R on $\mathbb{R}$ is $\{(a,b) \mid a \le b\}$

① ref? $\forall e\, (e \le e)$ true

② antisym? $\forall a \forall b\, (a \le b \wedge b \le a \rightarrow a = b)$ true

③ trans? $\forall a \forall b \forall c\, (a \le b \wedge b \le c \rightarrow a \le c)$ true

---

"Partial"?

(ex) R on $A = \{2, 3, 4, 6, 8, 10, 12\}$

$R = \{(a,b) \mid a \mid b\} = \{(a,b) \mid a \text{ is a factor of } b\}$

↑ such that
↑ divides

① ref? $e \mid e$ true

② antisym? $a \mid b \wedge b \mid a \rightarrow a = b$ ($a, b, \boxed{k_1, k_2}$ are all pos. ints)

$a \cdot k_1 = b \wedge b \cdot k_2 = a$

$a \cdot k_1 \cdot k_2 = a$ true

so $k_1 \cdot k_2 = 1 \rightarrow k_1 = k_2 = 1 \quad \therefore a k_1 = b$
$a = b$

(3) trans? $a|b \wedge b|c \to a|c$ true

So $(A, \text{divides})$ is a poset.

$A = \{2, 3, 4, 6, 8, 10, 12\}$

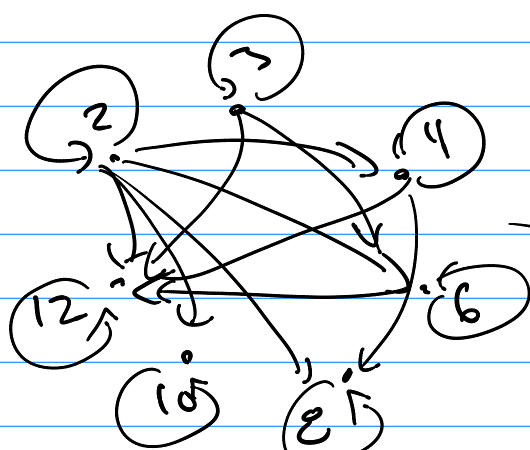
try it!

Compare 2 to 10	b/c	$2 \mid 10$	$2 \prec 10$
Compare 8 to 4	b/c	$4 \mid 8$	$4 \prec 8$
compare 12 to 3	b/c	$3 \mid 12$	$3 \prec 12$
compare 2 to 12	b/c	$2 \mid 12$	$2 \prec 12$
Compare 2 to 3	?!?	$2 \nmid 3$ and $3 \nmid 2$	

they can't be compared!

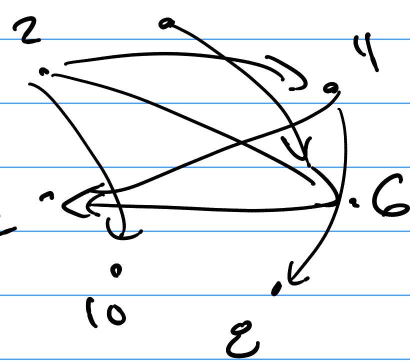
we call Posets partial orders b/c not everyone is comparable to everyone.

digraph:

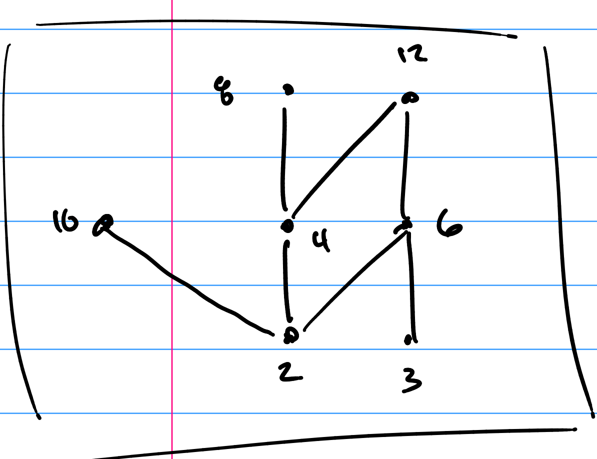


Simplify the digraph by not drawing anything that is obvious.

remove loops, trans



arrows up

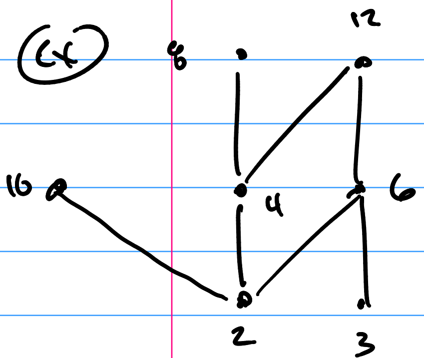


Hesse Diagram.

Application of Hasse Diagrams.

① **Maximal Elements** (have no element such that maximal ≺ e)

Ex



Ex) Maximals: { 10, 8, 12 }

② **Minimal Elements** (no e such that e ≺ minimal)

Ex) Minimals: { 2, 3 }

③ **greatest element**: one maximal? → greatest

③ **least element**: one minimal? → least