Math 322

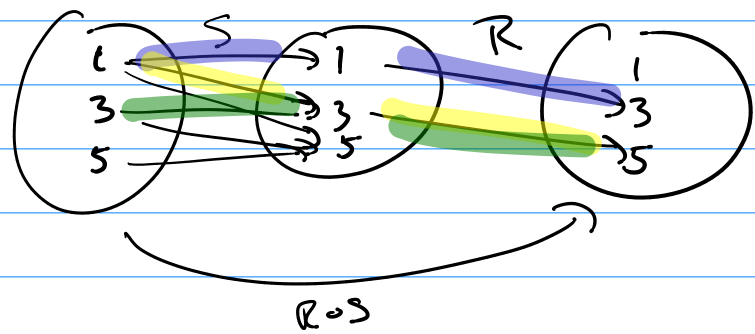
Q's 6.1 (2) R on $\{1,3,5\}$ S on $\{1,3,5\}$

set builder

$$R = \{(x,y) \mid y = x+2\} = \{(1,3),(3,5)\}$$

$$S = \{(x,y) \mid x \le y\} = \{(1,1),(1,3),(1,5),(3,3),(3,5),(5,5)\}$$

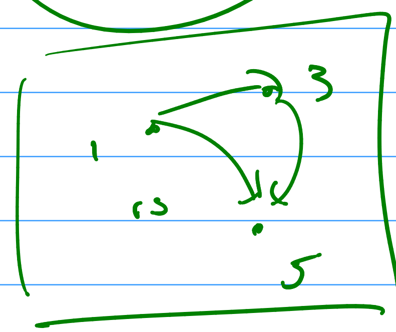
list: a) $R \circ S$ = rs ← textbook



$(1,3)$

$(1,5)$

$(3,5)$



Closures of Relations

Df: $A \subseteq B$

A: subset of B

B: superset of A

→ given R a relation on a set A.
if R $\boxed{\text{does not}}$ have a property, then can we make a new relation that is a superset of R that has the property?
Call it the $\boxed{\text{"property" closure of } R.}$

use R_c

(1) so $R \subseteq R_c$

R: does not have a property

R_c: does have the property

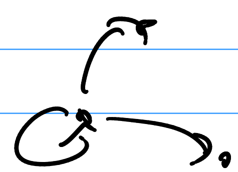
(2) add the minimal amount to R so the new R_c has the property.

What closures can happen?

(1) reflexive $\quad \forall c\,(c R c)$

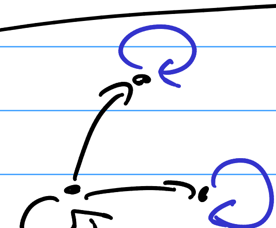
why?: b/c not reflexive means missing loops.

ex



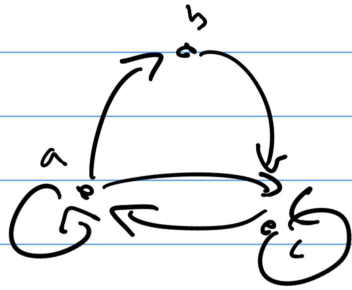
not reflexive

R

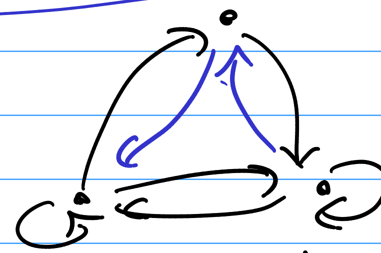


reflexive closure of R

② Symmetric $\forall a \forall b \, (aRb \to bRa)$



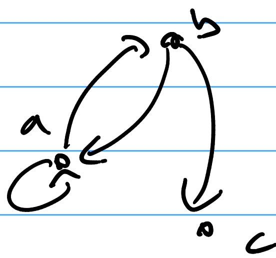
R is not symmetric



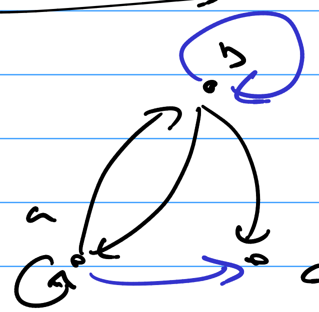
symmetric closure of R

③ transitive closure

$\forall a \forall b \forall c \, (aRb \wedge bRc \to aRc)$



R is not transitive

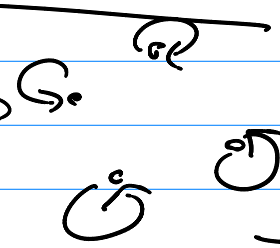


this new relation is transitive = transitive closure

Matrices for closures

① Reflexive Closure

$\boxed{R \cup ??}$



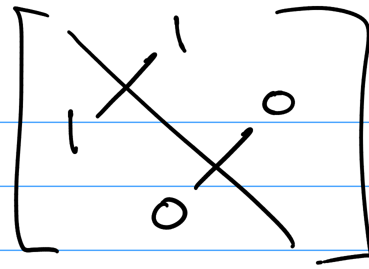
$\Delta = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$$M_{R \cup \Delta} = M_R \vee M_\Delta$$

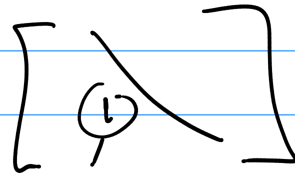
$$= M_R \vee \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

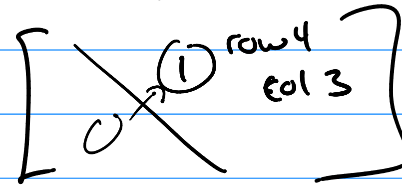
② Symmetric Closure

Symmetric



Know

$M_R =$ 

$M_R^T =$ 

1) $\boxed{M_{R \cup R^{-1}} = M_R \vee M_R^T}$

Transitive?

thm R is transitive iff $R^n \subseteq R$

Matrix notation:

R is transitive $\equiv$

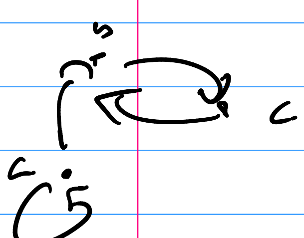
$M_R = M_R$

$M_R^{[2]} \subseteq M_R$

$M_R^{[3]} \subseteq M_R$

$M_R^{[4]} \subseteq M_R$

$\vdots$

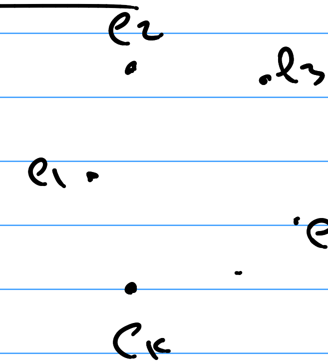


Know: $M_R^{[k]} = \underbrace{M_R \odot M_R \odot M_R \odot \cdots \odot M_R}_{k \text{ times}}$

So $M_R^{[k]}$ finds all (a,b) where a goes to b in a path of K steps.

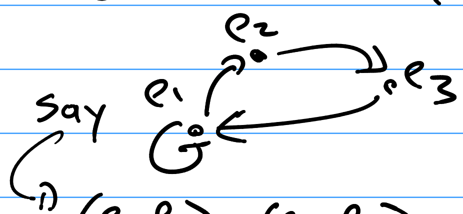
the transitive closure is ... R^*

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee \cdots$$



Note: R on set A and $|A| = K$

a path say



$(e_1, e_1), (e_1, e_2), (e_2, e_3)$

can be written as e_1, e_1, e_2, e_3

vs $\cancel{e_1, e_2, e_3, e_1,}\ e_1, e_2, e_3$

circle

e_1, e_2, e_3 shorter path that still connects e_1 to e_3

So by pigeonhole principle and understanding paths

transitive closure

$$\boxed{M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[|A|]}}$$

$|A| = K$ # of vertices

Warshal's Algorithm (next class)