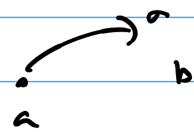
Math 322

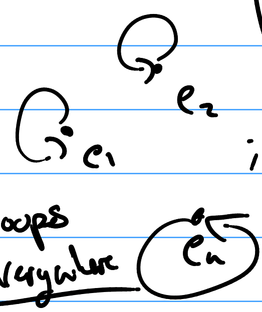
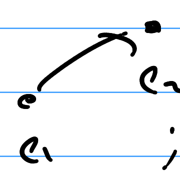
toys = Relations (from A to B) R is subset of $A \times B$

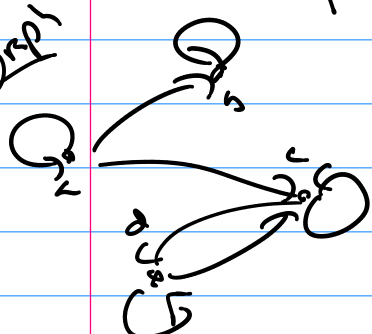
(or on a set A) R is subset of $A \times A$

aRb means $(a,b) \in R$

digraph  Matrix $M_R = \begin{array}{c} \\ a \end{array}\begin{array}{c} b \\ \begin{bmatrix} \cdots & 1 \end{bmatrix} \end{array}$

Properties

Name	Logic	digraph	Matrix
① Reflexive	$\dfrac{\forall e\,(eRe)}{A}$	 loops everywhere	$M_R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ only ones on main digonal
② Irreflexive	$\dfrac{\forall e\,\neg(eRe)}{A}$	 No loops anywhere	$M_R = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$

digraph 

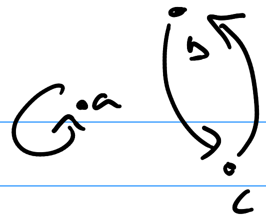
Matrix

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

R on insects

$R = \{ (a,b) \mid a \text{ eats } b \}$

(ex)



$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

reflexive? No

irreflexive? No

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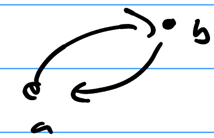
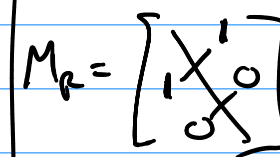
(3) Symmetric | $\forall a \forall b (aRb \to bRa)$



(If) you have an edge, you must have an edge back

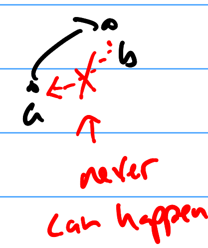


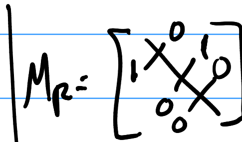
check?

$M_R^T = M_R$

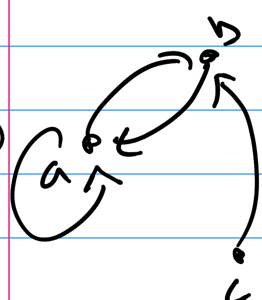
(4) Antisymmetric | $\forall a \forall b (aRb \land bRa \to a=b)$

$\equiv \forall a \forall b (a \neq b \to \neg(aRb \land bRa))$





(ex)



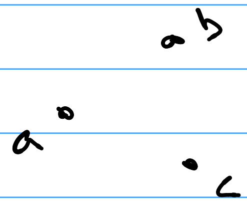
Reflexive? No (b/c $b \not R b$, $c \not R c$)

Irreflexive? No (b/c aRa)

Symmetric? No (b/c $cRb \land b \not R c$)

Antisymmetric? No (b/c $aRb \land bRa \land a \neq b$)

(ex)



Reflexive? No

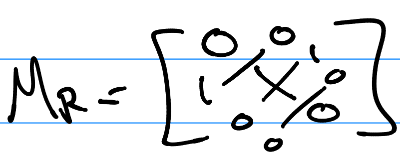
Irreflexive? Yes

Symmetric? Yes

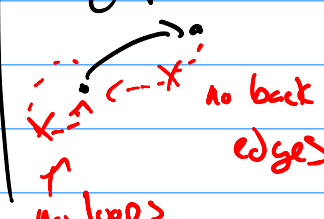
Antisymmetric? Yes

(5) Asymmetric $\equiv$ Irreflexive $\land$ Antisymmetric

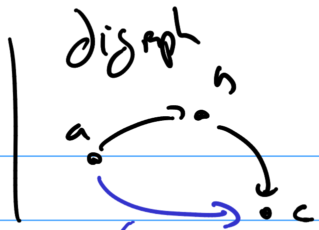
$\equiv \forall a \forall b (aRb \to b \not R a)$



digraph



⑥ Transitive | $\forall a \forall b \forall c\ (aRb \wedge bRc \rightarrow aRc)$ | digraph

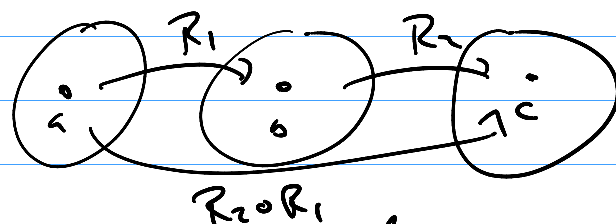


must have aRc

$$M_R = \begin{bmatrix} ? \end{bmatrix}$$

$\boxed{\text{thm}}$ R is transitive if and only if $\forall n\ R^n \subseteq R$, $n = 1, 2, 3, \ldots$

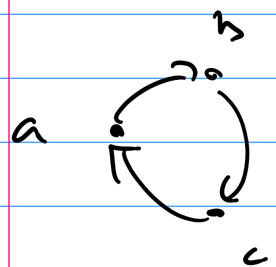
Def: Composition



$(R_2 \circ R_1)(a) = c$ or $R_2(R_1(a)) = c$

Power: $R^1 = R$

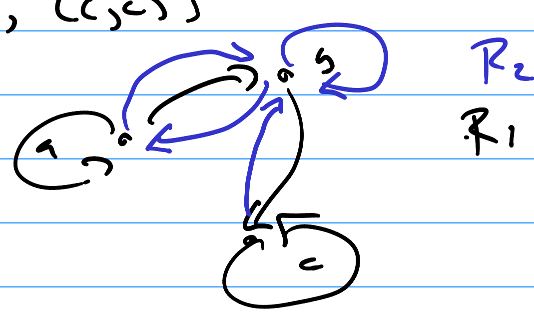
$R^n = R^{n-1} \circ R$ $\quad$ (or $R^n = \underbrace{R \circ R \circ R \circ \cdots \circ R}_{n\text{-times}}$)



$R = \{(a,b), (b,c), (c,a)\}$

$R^2 = R \circ R = 2 \text{ step} = \{(a,c), (b,a), (c,b)\}$

$R^3 = \{(a,a), (b,b), (c,c)\}$



using zero-one matrizes:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_{R_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So $M_{R^n} = M_R \odot M_R \odot \cdots \odot M_R = M_R^{[n]}$

$\boxed{\text{th}^m}$ R is transitive if and only if $\forall n\ R^n \subseteq R$ $n=1,2,3,\ldots$

$\longleftrightarrow$

PF $\boxed{\text{Case 1}}$ R is trans $\rightarrow$ $\forall n\ R^n \subseteq R$ $n=1,2,3,\ldots$

Idea: a) direct (assume R is trans) (show $\forall n\ R^n \subseteq R$)

b) induction for right side — use induction!

$\boxed{\text{Case 2}}$ $\forall n\ R^n \subseteq R$ $\rightarrow$ R is trans.

Idea: Univ. Instantiation for $\boxed{n=2}$ on direct proof.

Note: $R^2 \subseteq R$ is just a logical equiv of the def of transitivity.

Why? ① Equivalence Relations

→ find relationships that allow the statement...

"a is 'the same as' b"

② Partial Orderings (Posets)

- get to "total order"
- get to "well order"

R is an equivalence relation is
R is ① reflexive
② symmetric
③ transitive

$4 \mid n$?

$4 \mid 8 - -4$

mod 4

ex: $4 \equiv_4 0 \equiv_4 -4 \equiv_4 8$ etc

Ex $R = \{(a,b) \mid a \equiv_4 b\}$

Reflexive: $\forall e\,(eRe)$??

$e \equiv_4 e \to 4 \mid (e-e)$

$4 \mid 0$ true.

Symmetric $\forall a \forall b\,(aRb \to bRa)$???

$a \equiv_4 b \Rightarrow b \equiv_4 a$

$4 \mid (b-a) \leftrightarrow 4 \mid (a-b) \to b \equiv_4 a$

trans: (true)

$a \equiv b \quad b \equiv c$

$4 \mid (b-a) \quad 4 \mid (c-b)$

$\to 4 \mid (\cancel{b}-a) + (c-\cancel{b})$

$\to 4 \mid c-a$ so $a \equiv_4 c$

Posets