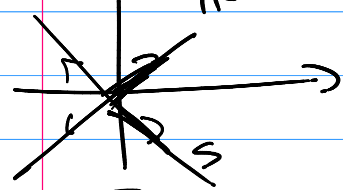
Math 511

$x^2 = y^2 \quad y = \pm x$

Q 3.1 1e) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; x_1^2 = x_2^2 \right\}$ subset of $\mathbb{R}^2$

$\mathbb{R}^2$



<u>check:</u>

1) $0 \in S$?
2) $V \in S$ is $\alpha V \in S$?
3) $V_1, V_2 \in S$ is $V_1 + V_2 \in S$?

S? ex $x_1^2 = x_2^2 \quad x_1 = \pm x_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \in S$

1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in S$ b/c $0^2 = 0^2$ or $0 = \pm 0$

2) if $\begin{bmatrix} a \\ \pm a \end{bmatrix} \in S$ is $\alpha \begin{bmatrix} a \\ \pm a \end{bmatrix} \overset{?}{\in} S$

$\begin{bmatrix} \alpha a \\ \pm \alpha a \end{bmatrix} \overset{?}{\in} S$ <u>true.</u>

3) <u>no</u> <u>Counter example</u> $\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$\quad \in S \qquad \in S \qquad \notin S$

$x_1^2 = x_2^2 \qquad 2^2 = 2^2$

$\Leftrightarrow$

$x_1 = \pm x_2 \qquad \begin{bmatrix} a \\ \pm a \end{bmatrix}$ ← basic element of S

Null Space $\quad N(A) = \{ x \mid Ax = 0 \}$

A is $m \times n$

ex $\quad A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$N(A) = ?$ $\quad$ all x such that $Ax = 0$

Solve:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \boxed{1} & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ \text{lead} & \text{free} & \text{free} & \text{lead} & \end{array}$$

$x_2 = \alpha$

$x_3 = \beta$

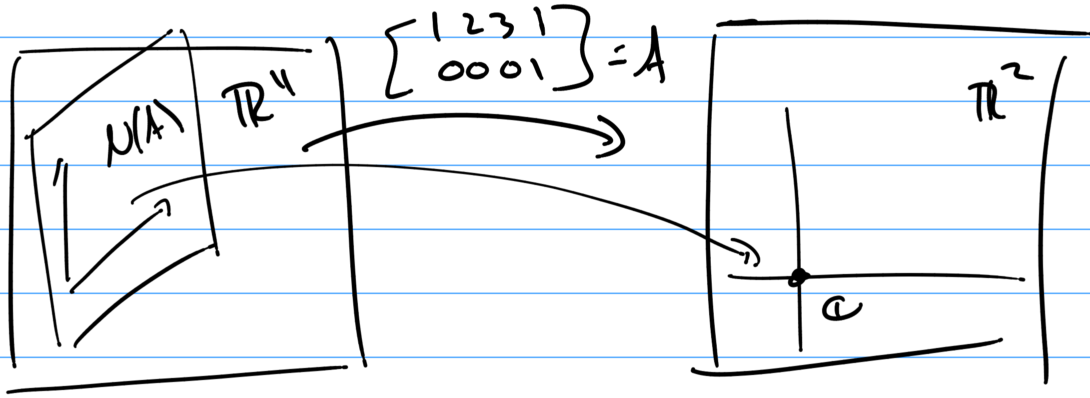
$x_4 = 0$

$x_1 = -2\alpha - 3\beta$

$$x = \begin{bmatrix} -2\alpha - 3\beta \\ \alpha + 0\beta \\ 0\alpha + \beta \\ 0\alpha + 0\beta \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N(A) = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Note:



Def: linear combination of $v_1, v_2, \dots, v_k$

$$\underbrace{\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k}} = v$$

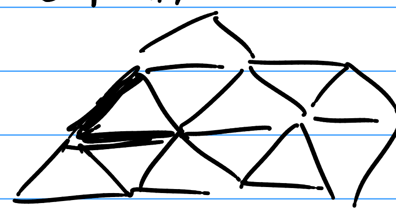
only uses our two vector space ops

$\alpha v, \quad v_1 + v_2$

Def: where can you all go with linear combo's

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$

set of all these for any α_i?



$\text{Span}(\{v_1, \dots, v_k\})$ = set of all linear combos of v_i

Thm $\text{Span}(\{v_i\})$ is a subspace of V.

$S = \text{Span}(\{v_i\})$ is a subspace of V

- say $v_1, v_2, \dots, v_k$ span S.
- say S is spanned by $v_1, v_2, \dots, v_k$
- if V, a vector space, is spanned by $v_1, v_2, \dots, v_k$ say the set $\{v_i\}$ is a spanning set of V.

How to show $v_1, v_2, \dots, v_k$ are a spanning set?

Solve: $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ variable representation of V

Q: do $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ span $\mathbb{R}^3$?

$$\alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

Solve

$\alpha_1 = \boxed{}$

$\alpha_2 = \boxed{}$

$\alpha_3 = \boxed{}$

Soln? → Spans

no Soln? → doesn't span

B Linear Independence

idea: dependence

$v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ $v_2 = \begin{bmatrix}-1\\1\end{bmatrix}$ $v_3 = \begin{bmatrix}1\\5\end{bmatrix}$

and I start talking about spanning sets and spans of v_1, v_2, v_3

any linear combination

Notice

$$2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$$

$$2v_1 + v_2 = v_3$$

$$2v_1 + v_2 - v_3 = 0$$

Def: linear dependence means a vector is a linear combo of the others.

If $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ has a non-trivial soln. → dependence

Def: if $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$
has only trivial soln → independence

Note: the primary definition of ind. vs dep.
is to solve $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$.

we can sometimes not actually solve it but
jump to the ans. of trivial vs non-trivial for
$\boxed{\text{some}}$ problems.

Ex are $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ linearly ind?

Long way, $\left[\begin{array}{ccc|c}1&1&1&0\\0&1&1&0\\0&0&1&0\end{array}\right] \rightsquigarrow$ solns?

Short way? We see this is $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

is of this form $\rightsquigarrow A\,x = 0$

$\det(A) = 1\cdot1\cdot1 = 1 \neq 0$
so only trivial soln → $\boxed{\text{ind.}}$

(ex) $C[a,b]$ Vector Space of Cont. functions over $[a,b]$

$C^{(k)}[a,b]$ Vector Space of Cont. functions over $[a,b]$ and whose $f', f'', f''', \ldots, f^{(k)}$ derivatives are also cont. over $[a,b]$

test.

$$\begin{cases} \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_k f_k = 0 \\ \alpha_1 f_1' + \alpha_2 f_2' + \cdots + \alpha_k f_k' = 0 \\ \vdots \\ \alpha_1 f_1^{(k-1)} + \cdots + \alpha_k f_k^{(k-1)} = 0 \end{cases}$$

$$\begin{vmatrix} f_1 & f_k \\ \vdots & \\ f_1^{k-1} & f_k^{k-1} \end{vmatrix}$$