Math 511

Section 3.6 Row Space, Column Space, (Fundamental Subspaces intro)

A is $m \times n$

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$ rect. set of scalars

$$A = [a_1 \ a_2 \ a_3 \ \cdots \ a_n]$$ Column vectors

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$$ row vectors

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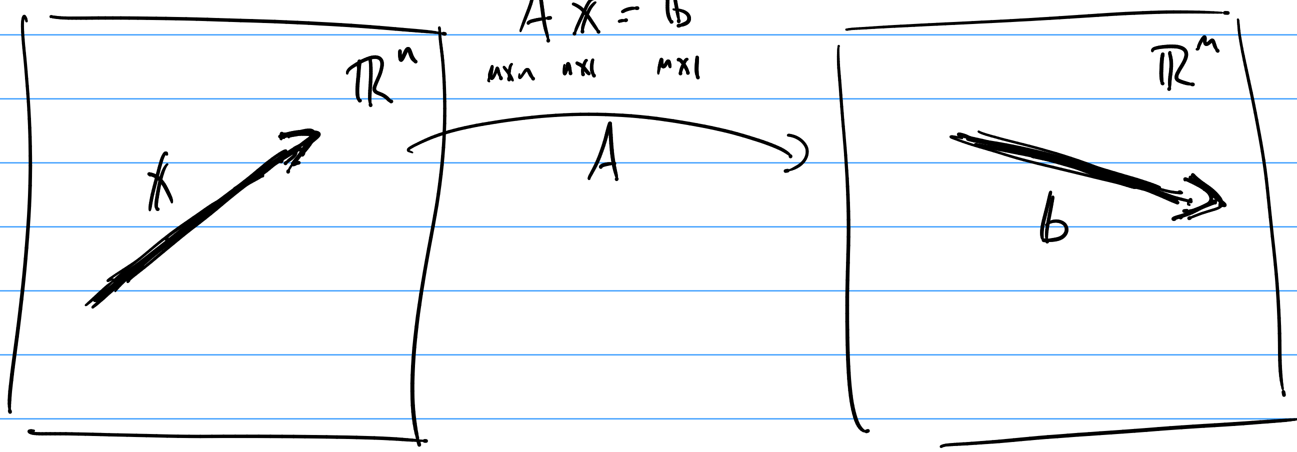
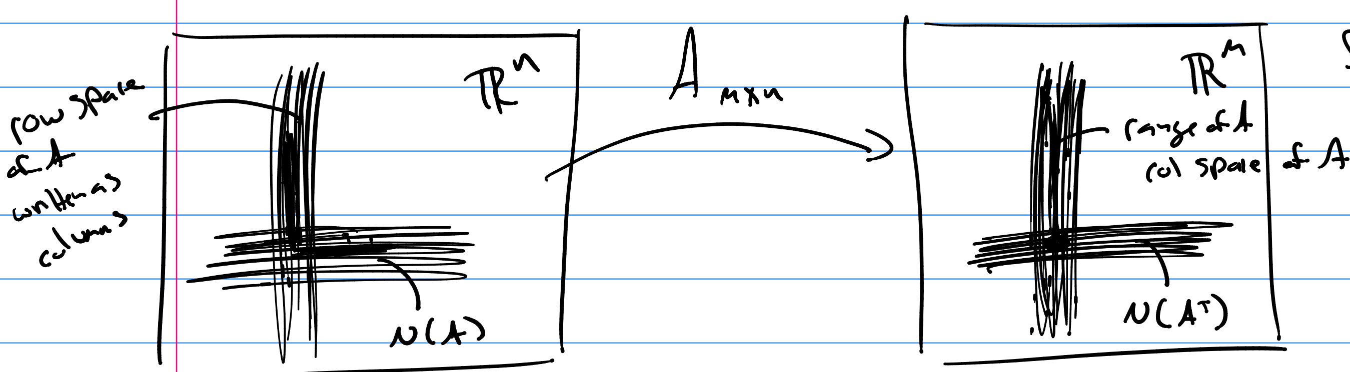
$$\underset{m \times n}{A}\ \underset{n \times 1}{x} = \underset{m \times 1}{b}$$



Solve: $Ax = b$ (find x given b)



Fundamental Subspaces (studied later)

3.6 $A = [a_1, a_2 \cdots a_n] = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

(1) row space of A is the span of $\vec{a}_i$:

row space of A = $\text{Span}(\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\})$

any such $\vec{v}$ ⇒ $\vec{v} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_m \vec{a}_m$

(2) Column space of A is the span of a_i:

column space of A = $\text{Span}(\{a_1, a_2, \ldots, a_n\})$

any such v → $v = \beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_n a_n$

(ex) $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

a) $\text{Span}(\{[1\ 2\ 1], [-1\ 0\ 1]\})$ = row space of A

any $\vec{v} = \alpha_1 [1\ 2\ 1] + \alpha_2 [-1\ 0\ 1]$

b) $\text{Span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}\right)$ = col space of A

any $v = \beta_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thm: If A and B are row equivalent then row space of A = row space of B.

ex $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 2 \\ -2 & -4 & -1 & -4 \end{bmatrix}$ going to do gauss-jordan elimination

$\xrightarrow[\text{equiv}]{\text{row}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{equiv}]{\text{row}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

$\xrightarrow[\text{equiv}]{\text{row}} \begin{bmatrix} \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{equiv.}]{\text{row}} \begin{array}{c} \left.\begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \\ \begin{matrix} \text{lead} & \text{free} & \text{lead} & \text{free} \end{matrix} \end{array}$ basis for row space

rank(A) = 2

rowspace of A = span({[1 2 1 3], [2 4 0 2], [-2 -4 -1, -4]})

by thm $\;=\; \boxed{\text{span}(\{[1\ 2\ 0\ 1], [0\ 0\ 1\ 2]\})}$

Def rank(A) = dim (row space of A)

example rank(A) = 2

Def rank(A) = # of lead vars after gauss-jordan elim.

Def dim(N(A)) = nullity of A

ex $\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 0 & 2 & 0 \\ -2 & -4 & -1 & -4 & 0 \end{array}\right] \xrightarrow[\text{ops}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -\alpha - 2\beta$

$x_3 = -2\alpha$

free free

$x_4 = \alpha$

$x_2 = \beta$

$\mathbf{x} = \begin{bmatrix} -\alpha - 2\beta \\ 0\alpha + 1\cdot\beta \\ -2\alpha + 0\beta \\ 1\alpha + 0\beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = N(A)$

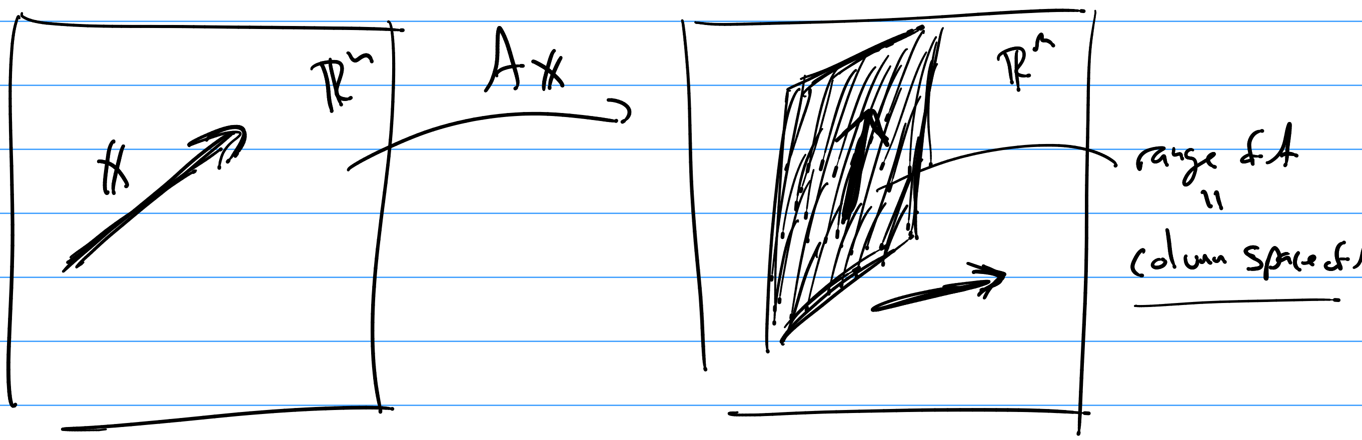
$\dim(N(A)) = 2$ = # of free vars

$\boxed{\text{Th}^m}$ $\underline{\text{rank}(A)} + \underline{\text{nullity}(A)} = n$

Column Space

Thm 1.3.1 $\boxed{\text{has}}$ $A\mathbf{x} = \mathbf{b}$ $\rightsquigarrow$ means $x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{b}$

$\boxed{\text{Th}^m}$ $A\mathbf{x} = \mathbf{b}$ is consistent iff $\mathbf{b} \in$ column space of A



$\boxed{\text{Th}^m}$ $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$

(1) iff column vectors span $\mathbb{R}^m$

$$(\text{Span}(\{\mathbf{a}_i\}) = \mathbb{R}^m)$$

(2) $A\mathbf{x} = \mathbf{b}$ has at most one solution for all $\mathbf{b} \in \mathbb{R}^m$

iff $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are linearly ind.

and not only that $(m = n)$

Corollary $A_{n\times n}$ is non-singular iff col vectors of A are a basis of $\mathbb{R}^n$

Now back to $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 2 \\ -2 & -4 & -1 & -4 \end{bmatrix} \xrightarrow[\text{gs}]{\text{row}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

gauss-jordan

<u>we found</u>: ① $\text{rank}(A) = 2$

② $\text{Nullity}(A) = 2$

③ $N(A) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$

④ row space of A = $\text{Span}([1\ 2\ 0\ 1], [0\ 0\ 1\ 2])$

⑤ col space of A

<u>reduced</u> <u>row echelon</u>

we can see the dependency of <u>U's</u> columns

$$\begin{array}{cccc} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \text{lead} & & \text{lead} & \\ P_1 & & P_2 & \end{array} \end{array}$$

<u>dependency equations</u>

$u_2 = 2u_1$

$u_4 = 1u_1 + 2u_3$

$\boxed{\text{th}^{\text{u}}}$ dependency equ's of U are same for A.

$a_2 = 2a_1 \leftarrow$

$a_4 = 1a_1 + 2a_3$

$$A = \begin{array}{cccc} & 2 & \text{dep} & \text{dep} \\ \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 2 \\ -2 & -4 & -1 & -4 \end{bmatrix} \\ 1 & & 2 & \end{array}$$

$$\boxed{\text{col space of } A = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)}$$

$\text{rank}(A) = \dim(\text{col space of } A)$