Math 511

Ch 3 Vector Spaces / Subspaces / basis / (ind/dep)

basis → linear independence
→ span the vector space

A very important thm

iff

$v_1, v_2, \ldots, v_k$ are linearly ind $\Longleftrightarrow$

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

the $c_1, c_2, \ldots, c_k$ are unique.

Hence any $v \in V$ and given a basis $v_1, v_2, \ldots, v_k$

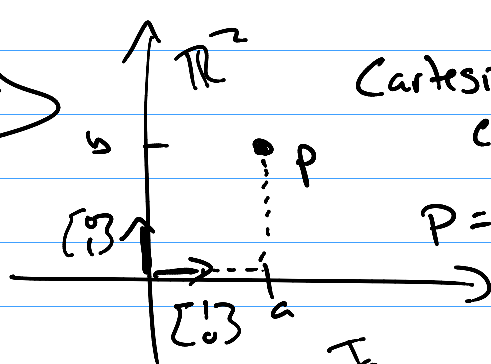
for $\dim(V) = k$

the c_i $\quad v = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ are uniq to v

So let those c_i be the coordinates of v

Concept

①



Cartesian coord.

$$P = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

123

$1 \cdot 10^2 + 2 \cdot 10 + 3 \cdot 1$

② 𝍸 IIII → I IX → 𝍸 𝍸

number

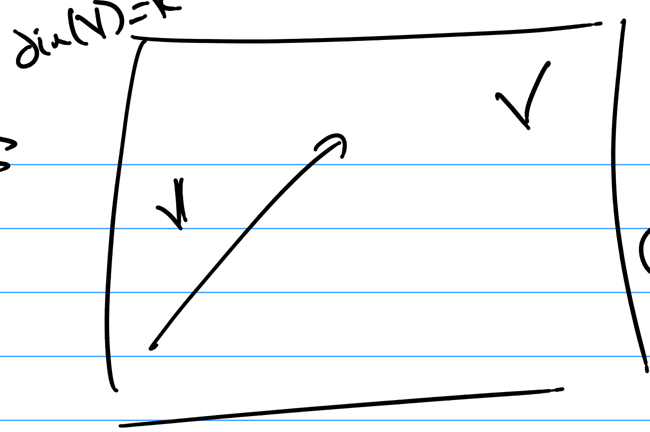
positional numbers

$b = 10$

$c_2(b)^2 + c_1 \cdot (b) + c_0 \cdot 1$

$(c_2, c_1, c_0)_{10} = (1, 2, 3)_{10}$

Vector Spaces



Basis: #1 $b_1, b_2, \dots, b_K$

#2 $d_1, d_2, \dots, d_K$

Notation: ① collect basis vectors

a) $B = [b_1 \ b_2 \ \dots \ b_K]$

b) $D = [d_1 \ d_2 \ \dots \ d_K]$

c) Standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$ ...

$$E = [e_1 \ e_2 \ \dots \ e_K] \quad \left(= I_{K \times K} \right)$$

② Coordinate notation

uniquely

$$v = c_1 b_1 + c_2 b_2 + \dots + c_K b_K = B \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix}$$

unique!

call these coordinates of v using basis B

Notation: $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix}$ ~~B~~

using basis B

So $[v]_D \rightarrow$ means $v = k_1 d_1 + k_2 d_2 + \dots + k_K d_K$

$$[v]_D = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_K \end{bmatrix}_D$$

uniq to d_i

Similar idea to $(1,1,0)_2 = \text{𝍸 } 1 = (1,1)_5 = (6)_{10}$

4 2 1 5 1

③ <u>Standard basis</u> $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\end{bmatrix}, \ldots$

$\mathbb{R}^3$ (ex) $$\begin{bmatrix}a\\b\\c\end{bmatrix}_E = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$$

"typically" in notation $[v]_E$ is also $[v]$ ◯ ← nothing here

Basis Conversions

how to go from basis $\mathcal{B}$ coordinates to basis $\mathcal{D}$ coordinates.

$$\begin{tikzcd} & {[v]_B} \arrow[d, bend left, "?"] \\ v \arrow[ur] \arrow[r] & {[v]_D} \end{tikzcd}$$

① Convert from a non-standard basis to standard.

1st fact $$\begin{bmatrix}a\\b\\c\end{bmatrix}_E = a\,e_1 + b\,e_2 + c\,e_3$$

2nd fact $$[v]_B \rightarrow \boxed{c_1 b_1 + c_2 b_2 + \cdots + c_k b_k} \rightarrow \begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix}_B$$

$$v = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = \boxed{B\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix}_B}$$

together:

$$\boxed{[v]_E = B\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix}_B}$$

← Convert base $\mathcal{B}$ to standard.

Ⓐ to convert base B coord to standard ...

$$[v]_E = B\,[v]_B$$

b/c B is storing a <u>basis</u> → says B^{-1} exists

so $B^{-1}[v]_E = B^{-1}B\,[v]_B$

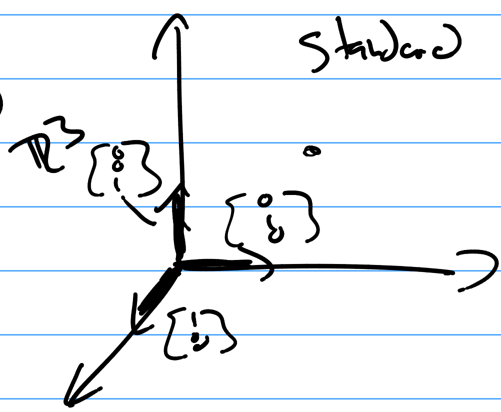
Ⓑ Convert from standard to non-standard basis

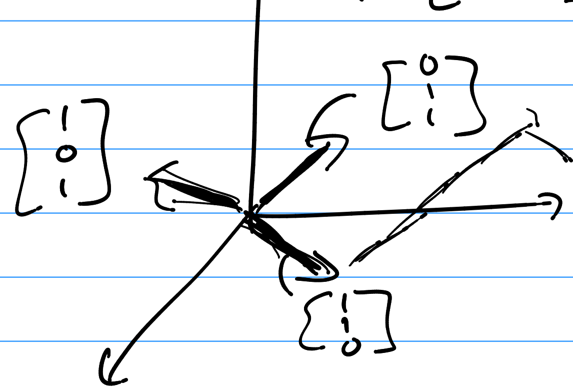
$$[v]_B = B^{-1}\,[v]_E$$

standard

$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

ex $\mathbb{R}^3$





$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_E = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}_E$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_E = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_B$$

$B \qquad\qquad D$

Non-standard to non-standard

$$B[V]_B = [V]_E = D[V]_D$$

$$B[V]_B = D[V]_D$$

$$\therefore \quad [V]_B = \overbrace{B^{-1}D}^{S^{-1}}[V]_D \qquad (D \text{ to } B \text{ coord})$$

$$[V]_D = \underbrace{D^{-1}B}_{S}[V]_B \qquad (B \text{ to } D \text{ coord})$$