Math 511

Subspaces

1. Vectorspace (variables)
2. ops & Vectorspace
3. $\mathbf{0}$ & Vectorspace
4. Subset & Vectorspace (variables)

check if Subspace

3 tests
- a) $\mathbf{0} \in S$
- b) $\alpha v_1 \in S$
- c) $v_1 + v_2 \in S$

3.2 #1

ex $\mathbb{R}^2$ $\quad S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \; x_1^2 = x_2^2 \right\}$

1. any $\begin{bmatrix} a \\ b \end{bmatrix}$
2. normal vector ops
3. $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

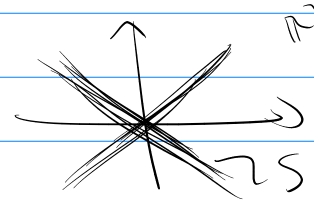
S 4. $\begin{bmatrix} c \\ d \end{bmatrix}$ such that $c^2 = d^2$

or $c = d$, $c = -d$

all prelim information

ex $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

Visual



3.2 5c

1. P_4 (Vectorspace) $\quad p = a + bx + cx^2 + dx^3$
2. ops: αp, $p_1 + p_2$ are normal poly. ops
3. $\mathbf{0} = 0 + 0x + 0x^2 + 0x^3$
4. Subset (5c) $\quad$ S is all polynomials in P_4 such that $p(0) = 0$

ex $\quad p_1 = 3 + 2x - 0x^2 + x^3 \quad p_1(0) = 3 \notin S$

$p_2 = 0 + 2x - x^2 + x^3 \quad p_2(0) = 0 \in S$

so p in S is $\boxed{p = 0 + bx + cx^2 + dx^3}$

Subspace: 1. is $\mathbf{0} \in S$?

$z(x) = 0 + 0x + 0x^2 + 0x^3$ in $0 + bx + cx^2 + dx^3$ true

$z(0) = 0$

② $\alpha p = \boxed{0} + \alpha b x + \alpha c x^2 + \alpha d x^3$ in S

true

③ $p_1 + p_2 =$

$\boxed{0} + b_1 x + c_1 x^2 + d_1 x^3 + \boxed{0} + b_2 x + c_2 x^2 + d_2 x^3$

in S

true

So it is a subspace

3.3 Linear Ind.

Solve $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0$ ←

① only trivial soln → ind. —

② non-trivial soln → dep ←

3.3 4c $\mathbb{R}^{2\times 2}$ — vector space is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ space

are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ independent

"solve" $\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

"technique" ① guess $2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

so non-trivial soln! → dependent!

(2) ~~non-guess~~ $\begin{bmatrix} \alpha_1 + 2\alpha_3 & \alpha_2 + 3\alpha_3 \\ 0 & \alpha_1 + 2\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\alpha_1 + 2\alpha_3 = 0$
$\alpha_2 + 3\alpha_3 = 0$
~~$\alpha_1 + 2\alpha_3 = 0$~~

$\alpha_1 + 2\alpha_3 = 0$
$\alpha_2 + 3\alpha_3 = 0$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$

free so non-trivial soln
so $\boxed{\text{dep}}$

3.3 #7 hold X_1, X_2, X_3 are linearly ind.

($\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$ has only $\alpha_1 = \alpha_2 = \alpha_3 = 0$ trivial soln)

check Y_1, Y_2, Y_3

where $Y_1 = X_2 - X_1$
$Y_2 = X_3 - X_2$
$Y_3 = X_3 - X_1$

linearly ind/dep?

guess?

$Y_1 + Y_2 = (X_2 - X_1) + (X_3 - X_2)$
$= X_3 - X_1 = Y_3$

$(1)Y_1 + (1)Y_2 + (-1)Y_3 = 0$

$\boxed{\text{dep!}}$

"Solve"

go back to $\boxed{\beta_1 V_1 + \beta_2 V_2 + \beta_3 V_3 = 0}$

$$\beta_1 (X_2 - X_1) + \beta_2 (X_3 - X_2) + \beta_3 (X_3 - X_1) = 0$$

$$\beta_1 X_2 - \beta_1 X_1 + \beta_2 X_3 - \beta_2 X_2 + \beta_3 X_3 - \beta_3 X_1 = 0$$

$$(-\beta_1 - \beta_3)\boxed{X_1} + (\beta_1 - \beta_2)\boxed{X_2} + (\beta_2 + \beta_3)\boxed{X_3} = 0$$

b/c X_1, X_2, X_3 are LI, this has only trivial sol.

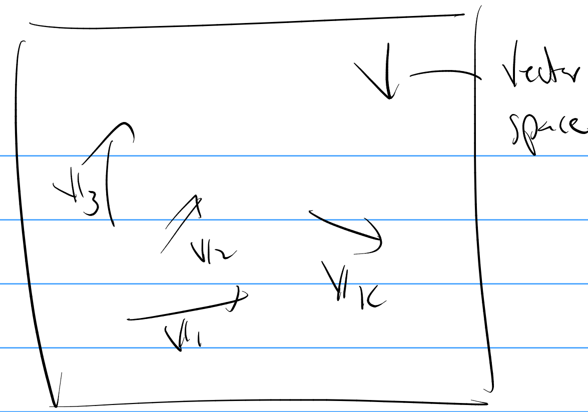
$$\begin{aligned} -\beta_1 - \beta_3 &= 0 \\ \beta_1 - \beta_2 &= 0 \\ \beta_2 + \beta_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} \beta_2 - \beta_1 &= 0 \\ -\beta_2 + \beta_1 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \boxed{-1} & 0 & -1 & 0 \\ 0 & \boxed{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free so $\boxed{\text{dep}}$

Basis & Dimension

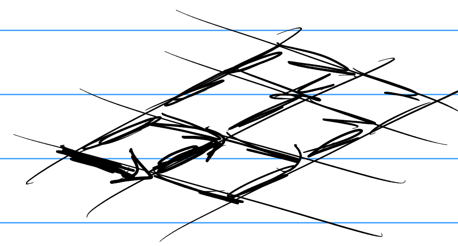


$\mathrm{Span}(v_1, v_2, \ldots, v_k)$

all $v = \alpha_1 \underline{v_1} + \alpha_2 \underline{v_2} + \cdots + \alpha_k \underline{v_k}$

2D "look"

$\mathbb{R}^2$



if $\underbrace{\mathrm{Span}(v_1, v_2, \ldots, v_k)}_{\text{spanning set}} = V$

What is uniq. about a space is $\boxed{\text{not}}$ the vectors that span it but how many are needed.

$\rightarrow$ dimension of V

ex

$\mathbb{R}^2$ $\quad \dim(\mathbb{R}^2) = 2$

$\mathbb{R}^5$ $\quad \dim(\mathbb{R}^5) = 5$

$\mathbb{R}^{2\times 7}$ $\quad \dim(\mathbb{R}^{2\times 7}) = \underline{14}$

need this many to span and they must be independent.

Consider $\mathbb{P}$ (poly space) any polynomial

$p = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots$

dim(P) = not finite call $\boxed{\text{dim}(P) \text{ is infinite}}$

assume dim(P) = k (a finite number)

so $\boxed{p_1, p_2, \dots, p_k}$ — use for a basis if independent

also: $p_1, p_2, \dots, p_k, p_{k+1}$ must be dependent

use wronskian on $p_1 = 1,\ p_2 = x,\ p_3 = x^2, \dots, p_{k+1} = x^k$

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^k \\ 0 & 1 & 2x & \cdots & kx^{k-1} \\ 0 & 0 & 2 & \cdots & k(k-1)x^{k-2} \\ \vdots & & & & \\ 0 & 0 & \cdots & & k! \end{vmatrix} = 0! \cdot 1! \cdot 2! \cdot 3! \cdots k!$$

$\neq 0$

says ind.