Math 511

$$\text{Col. Space} = \text{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) = \text{Span}(\text{only ind. col's})$$

ind. cols

A's col.

$$\text{Row Space} = \text{Span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m) = \text{Span}(\text{only ind. rows of } U)$$

A's row's

keep independent ones

$$A = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & -4 & 1 & 4 \\ 3 & -6 & -2 & 7 \end{bmatrix} \xrightarrow[\text{red. row ech.}]{\text{row ops}} U = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ind. ind.

R_1 R_2

lead lead

free free

dependent columns

$$\boxed{\begin{aligned} \vec{a}_2 &= -2\vec{a}_1 \\ \vec{a}_4 &= 1\cdot\vec{a}_1 + 2\vec{a}_3 \end{aligned}}$$

dep. eqn's

$$\begin{aligned} \vec{u}_2 &= -2\vec{u}_1 \\ \vec{u}_4 &= 1\vec{u}_1 + 2\vec{u}_3 \end{aligned}$$

$$\text{Col Space of } A = \boxed{\text{Span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\end{bmatrix}\right)} \rightsquigarrow C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\text{Row space of } A = \boxed{\text{Span}([1\ -2\ 0\ 1], [0\ 0\ 1\ 2])} \rightsquigarrow R = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Note:

$$\underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & -2 \end{bmatrix}} \underset{2\times 4}{\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & -2 & -1 & -1 \\ 2 & -4 & 1 & 4 \\ 3 & -6 & -2 & -7 \end{bmatrix}$$

$$CR = A$$

Hey!

$$A = LU$$

now $A = CR$

36 #6 $\boxed{A\mathbf{x} = \mathbf{b}}$ ~ solve? Fact

~ 1 soln } consistent
~ ∞ solns
~ ~~0~~ solns inconsistent

how many solns if $\mathbf{b}$ is in col. space of A

and A's col's are linearly dep.?

(1) $\mathbf{b}$ in col space of A says there are $c_1, c_2, \ldots, c_k$

such that $\boxed{c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \cdots + c_k \mathbf{a}_k = \mathbf{b}}$

$A\mathbf{x} = \mathbf{b}$, $\mathbf{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ or $A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \mathbf{b}$ or there is a soln!

thm 3.6.2 p. 156

So 0 soln's is out and we are either 1 or ∞

(2) Next: Add A's col's are linearly dep.

So we have free var's so $\boxed{\infty \text{ soln's}}$

3.6 #10

A is $m \times n$ so $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ — m-rows

n-cols

n dotted variables

rank(A) = n so n-lead variables

$\therefore$ 0 free variables

if $A\mathbb{c} = A\mathbb{d} = \mathbb{y}$ w/ consistent

$A \rightsquigarrow U = \begin{bmatrix} 1 & 2 & 4 & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

5×3

rank(A) = 3

b/c no-free ↗ only 1 soln

$m > n$

$A\mathbb{c} = A\mathbb{d} = \mathbb{y}$

only one soln → $\boxed{\mathbb{c} = \mathbb{d}}$

$A\mathbb{c} = \mathbb{y}$ $\quad$ $A\mathbb{d} = \mathbb{y}$

Exam 2 $\quad$ 11 probs @ 10pts = 110pts

but 100pts = 100%

(1) is V a vector space $\begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

objects are either $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^{2\times 2}$

ops scalar mult. $\alpha \odot \mathbb{v}$

addition $\mathbb{v}_1 \oplus \mathbb{v}_2$

ex $\mathbb{R}^{2\times 2}$ define $\alpha \odot \mathbb{v}$ is $\alpha \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & b \\ c & \alpha d \end{bmatrix}$

define $\mathbb{v}_1 \oplus \mathbb{v}_2$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+f & b+e \\ c+h & d+g \end{bmatrix}$

check (1) (2)

A1) to A8)

check all

V can be $\mathbb{R}^{2\times2}$, $\mathbb{R}^2$, or P_3

(2)
(3) } is S a subspace

(ex) for $\mathbb{R}^{2\times2}$ and $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

diagonal matrix.

Subspace?

check: A3) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \overset{?}{\in} S$ true

C1 (2) is $2\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \overset{?}{\in} S$ $\begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix} \in S$ true

C2 (3) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in S$ true

Subspace!

(4)
(5) } given $v_1, v_2, \dots, v_k$ & V — $C[a,b]$, $\mathbb{R}^3$, P_3

check linear ind.

Hint: if I give you v_1, v_2, v_3, v_4, v_5 of $\mathbb{R}^3$

Do you $\boxed{\text{need}}$ to check?

3:3 $\boxed{2c}$ $\mathbb{R}^3$ ind: $|\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0|$

with v_1, v_2, v_3

so $A = \underbrace{[v_1\ v_2\ v_3]}_{3}$ is $\boxed{\text{Square}}$

use $\det(A) = 0 \Rightarrow$ dep, $\neq 0 \Rightarrow$ ind.

4c $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 3 & 4 \end{bmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8-3 = 5 \neq 0$ ind.

6, 7 Basis Dimension $\mathbb{R}^n$

6 $\dim(\text{Span}(v_1, v_2, \dots, v_k)) = \#$ of ind. v_i

7 ind / dep in P_n $\boxed{S}^{4} = \mathbb{R}^n$

8, 9 change of basis

8 $[x]_D = \boxed{D^{-1}B}[x]_B$

know $[x]_E = B[x]_B$

$[x]_B = B^{-1}[x]_E$

9 basically same but P_n

10, 11 See top! (and) $N(A) \to [A|0]$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \to x_3 = -2\beta \\ \to x_1 = 2\alpha - \beta \end{array} \quad x = \begin{bmatrix} 2\alpha - \beta \\ \alpha \\ -2\beta \\ \beta \end{bmatrix}$$

free free

$x_2 = \alpha$ $x_4 = \beta$

$$x = \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$N(A) = \text{Span}\left(\begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-2\\1\end{bmatrix}\right)$$

$$= \text{Span}\left([2\ 1\ 0\ 0]^T, [-1\ 0\ -2\ 1]^T\right)$$

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ex for #1

$\mathbb{R}^{2\times 2}$

space

define $\alpha \odot V_1$ is $\alpha \odot \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\alpha a & b\\ c & \alpha d\end{bmatrix}$

define $V_1 \oplus V_2$ is $\begin{bmatrix}a & b\\ c & d\end{bmatrix} \oplus \begin{bmatrix}e & f\\ g & h\end{bmatrix} = \begin{bmatrix}a+f & b+e\\ c+h & d+g\end{bmatrix}$

$\mathbb{R}^{2\times 2}$ so objects $= \begin{bmatrix}a & b\\ c & d\end{bmatrix}$

use $\odot$ $\alpha V_1 = \alpha \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\alpha a & \alpha b\\ \alpha c & \alpha d\end{bmatrix}$

given

$\oplus \to \begin{bmatrix}a & b\\ c & d\end{bmatrix} \oplus \begin{bmatrix}e & f\\ g & h\end{bmatrix} = \begin{bmatrix}a+f & b+e\\ c+h & d+g\end{bmatrix}$

Axioms:

(1) is $\begin{bmatrix}\alpha a & \alpha b\\ \alpha c & \alpha d\end{bmatrix} \in \mathbb{R}^{2\times 2}$ Yes

(2) is $\begin{bmatrix}a+f & b+e\\ c+h & d+g\end{bmatrix} \in \mathbb{R}^{2\times 2}$ Yes

A1) Commutative $V_1 \oplus V_2 = V_2 \oplus V_1$

$$V_1 + V_2 = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \oplus \begin{bmatrix}e & f\\ g & h\end{bmatrix} = \begin{bmatrix}a+f & b+e\\ c+h & d+g\end{bmatrix}$$

$$V_2 + V_1 = \begin{bmatrix}e & f\\ g & h\end{bmatrix} \oplus \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}e+b & f+a\\ g+d & h+c\end{bmatrix} \neq$$

(c+2)

objects: $\mathbb{R}^3$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

ops ① Scalar Mult. αv_1 is $\alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}$

② addition $v_1 + v_2$ is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$

(1) $\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix} \overset{?}{\in} \mathbb{R}^3$ yes

(2) $\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} \overset{?}{\in} \mathbb{R}^3$ yes

A1) Commutativity $v_1 + v_2 = v_2 + v_1$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$$

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d+a \\ e+b \\ f+c \end{bmatrix}$$

|| yes

A2) Assoc. $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \left(\begin{bmatrix} e \\ f \\ g \end{bmatrix} + \begin{bmatrix} h \\ i \\ j \end{bmatrix} \right) = \begin{bmatrix} a+(e+h) \\ b+(f+i) \\ c+(g+j) \end{bmatrix}$$

$$= \begin{bmatrix} (a+e)+h \\ (b+f)+i \\ (c+g)+j \end{bmatrix}$$

|| yes

A3) $\mathbb{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ identity $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ true

A4) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \underbrace{\begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}}_{\text{inv.}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓ Yes

A5) etc

$\vdots$

A8)