Math 511 - Exam 2

0) Write your name and MyWSUid:

0) Write the time you started the exam:

scalar mult. vector addition

$\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix} = \boxed{\begin{bmatrix}\alpha x_1\\x_2\end{bmatrix}}; \quad \begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}y_1\\y_2\end{bmatrix} = \boxed{\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}}$

1) For the set of vectors in $\mathbb{R}^2$ define addition normally but scalar multiplication by $\alpha \boldsymbol{x} = [\alpha x_1, x_2]^T$. Check if all 10 axioms apply. And is this a vector space?

(1) $\begin{bmatrix}\alpha x_1\\x_2\end{bmatrix}$ is in $\mathbb{R}^2$ (2) $\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}$ is in $\mathbb{R}^2$

✓ A1) $\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix} = \begin{bmatrix}y_1+x_1\\y_2+x_2\end{bmatrix}$ b/c commutative prop. of scalars

✓ A2) $\begin{bmatrix}x_1+(y_1+z_1)\\x_2+(y_2+z_2)\end{bmatrix} = \begin{bmatrix}(x_1+y_1)+z_1\\(x_2+y_2)+z_2\end{bmatrix}$ b/c assoc. prop. of scalars

✓ A3) $\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ so $\begin{bmatrix}0\\0\end{bmatrix} = \mathbf{0}$ of $\mathbb{R}^3$

✓ A4) $\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}-x_1\\-x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ so $\begin{bmatrix}-x_1\\-x_2\end{bmatrix}$ is additive inv.

✓ A5) $\alpha\begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix} = \begin{bmatrix}\alpha x_1+\alpha y_1\\x_2+y_2\end{bmatrix}$

$\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}+\alpha\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}\alpha x_1+\alpha y_1\\x_2+y_2\end{bmatrix}$ } equal

✗ A6) $(\alpha+\beta)\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}(\alpha+\beta)x_1\\x_2\end{bmatrix}$

$\alpha\begin{bmatrix}x_1\\x_2\end{bmatrix}+\beta\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}(\alpha+\beta)x_1\\2x_2\end{bmatrix}$ $\neq$ not equal

(so $\boxed{\text{not}}$ a vector space)

✓ A7) $\begin{bmatrix}\alpha\beta x_1\\x_2\end{bmatrix} = \alpha\begin{bmatrix}\beta x_1\\x_2\end{bmatrix}$

✓ A8) $1\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

but

2) Consider the set of polynomials $p(x)$ in P_3 such that $p(x)$'s graph is a parabola (these are all $p(x) = ax^2 + bx + c$ where a is never zero). Is this set a subspace of of P_3? Explain.

S

① $0 = 0x^2 + 0x + 0$ is not in the subset ...

b/c a is zero

② if $\alpha = 0$ $\quad 0(2x^2+1) = 0x^2 + 0x + 0 \notin S$

in S, not in S

not closed

③ $(-x^2+1) + (x^2+1) = 2 \notin S$

in S, in S, not in S

not closed

Not a subspace

3) Does the set of all 2 x 2 matrices A such that $a_{1,2} = 0$ form a subspace of $\mathbb{R}^{2\times 2}$? Explain.

$\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$

① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ true

② $\alpha \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{bmatrix} \in S$ true

③ $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} + \begin{bmatrix} d & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ b+e & c+f \end{bmatrix} \in S$ true

so Subspace ✓

$\mathbb{R}^3$

4) Let $\boldsymbol{x}_1 = [1, 0, 1]^T, \boldsymbol{x}_2 = [2, 4, 0]^T$, and $\boldsymbol{x}_3 = [0, 4, -2]^T$. Are the vectors linearly independent? Prove your answer.

$$\det(X) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 4 \\ 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 0 & -2 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix}$$

$$= -8 + 8 = 0$$

Not linear ind.

$\to$ dependent.

5) Are $\{(1), (2 + x), (x + x^2), (x^2 + 1), (1 + x + x^3\}$ linearly independent in P_4? Prove your answer.

No. $\dim(P_4) = 4$ and we have 5 vectors.

6) Consider the vectors $\boldsymbol{x}_1 = [1, 2, 1]^T, \boldsymbol{x}_2 = [2, 5, 0]^T, \boldsymbol{x}_3 = [1, 3, -1]^T, \boldsymbol{x}_4 = [-2, -4, -2]^T$, and $\boldsymbol{x}_5 = [4, 9, 2]^T$. What is the dimension for the Span of the vectors?

$$\begin{bmatrix} 1 & 2 & 1 & -2 & 4 \\ 2 & 5 & 3 & -4 & 9 \\ 1 & 0 & -1 & -2 & 2 \end{bmatrix} \xrightarrow[\text{RO}]{\text{RO}} \begin{bmatrix} 1 & 2 & 1 & -2 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 & -2 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So only x_1, x_2 are ind.

So $\dim(\mathrm{Span}(x_i)) = 2$

$\left[\text{basis} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \right]$

$a + bx + cx^2$ is $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

7) Create three linearly independent polynomials from P_3 (and DO NOT choose the three standard basis polynomials for P_3 and make sure that none of the coefficients in your polynomials are a 1). Verify your polynomials are independent and that they form a basis for P_3.

need 3 independent

So $\begin{bmatrix} 2 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & \pi \end{bmatrix}$ & $\{2, 3+2x, 2+\pi x^2\}$ ✓

has det $\begin{pmatrix} 2 & 3 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & \pi \end{pmatrix} = 4\pi \neq 0$ so ind.

$\boxed{\text{or } c_1(2) + c_2(3+2x) + c_3(2+\pi x^2) = 0 + 0x + 0x^2 \text{ so solve and show } c_1 = c_2 = c_3 = 0 \text{ is only soln.}}$

8) Find the transition matrix to change vectors in basis $\boldsymbol{b}_1 = [1,1,1]^T$, $\boldsymbol{b}_2 = [0,1,1]^T$, and $\boldsymbol{b}_3 = [0,0,1]^T$ into the basis $\boldsymbol{d}_1 = [1,0,1]^T$, $\boldsymbol{d}_2 = [0,2,3]^T$, and $\boldsymbol{d}_3 = [0,1,2]^T$.

$$[x]_D = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{D^{-1}\ B} [x]_B$$

$S = D^{-1}B$

D^{-1} ?

So
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 & 1 & 1 \end{array}\right]$$

r_2 swap r_3 then $r_2 - r_3$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 & -1 & 2 \end{array}\right]$$

I $D^{-1}B$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ -3 & -1 & 2 \end{bmatrix}.$$

a + bx is $\begin{bmatrix} a \\ b \end{bmatrix}$

9) For P_2 with bases $B = \{1+x, 1+2x\}$ and $D = \{1+3x, 2+7x\}$, find the matrix, and call it S, representing the change of base from B to D. Find the transition matrix, and call it T, representing the change of base from D to B.

$B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ so $S = D^{-1}B$ $T = S^{-1} = B^{-1}D$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 3 & 7 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$D^{-1}B$

$S = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 5 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

$B^{-1}D$

$T = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$

(10) You take a + bx is $\begin{bmatrix} b \\ a \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 3 & 7 \\ 1 & 2 \end{bmatrix}$ $\to$ $S = D^{-1}B = ?$ $T = B^{-1}D = ?$ is also correct.

10) Let A be the given 3×5 matrix ...

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 1 & 5 & 10 \\ -1 & -2 & 1 & -1 & -2 \end{pmatrix}$$

a) use row operations on A to create U in reduced row eschelon form — zeros below leads, zeros above leads

a) determine the rank(A), the nullity of A, and the dependency equations.

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 1 & 5 & 10 \\ -1 & -2 & 1 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} \boxed{1} & 2 & 0 & 2 & 4 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$\boxed{\text{rank}(A) = 2}$ — leads

nullity$(A) = 5 - 2 = \boxed{3 = \text{Nullity}(A)}$ — frees

A's $\to$ $\boxed{a_2 = 2a_1}$, $\boxed{a_4 = 2a_1 + a_3}$, $\boxed{a_5 = 4a_1 + 2a_3}$

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 1 & 5 & 10 \\ -1 & -2 & 1 & -1 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11) Using your work from 10 ...

a) write N(A) as a Span.

b) write the column space of A as a Span and as a matrix C.

c) write the row space of A as a Span and as a matrix R.

d) verify $CR = A$.

b) col space = $\text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$, $C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}$

c) row space = $\text{Span}\left([1\ 2\ 0\ 2\ 4], [0\ 0\ 1\ 1\ 2]\right)$, $R = \begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 4 & 1 & 5 & 10 \\ -1 & -2 & 1 & -1 & -2 \end{bmatrix}$

a) N(A)

$$\begin{array}{c} U \\ \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

free

$x_1 = -2x_2 - 2x_3 - 4x_4$

$x_3 = \quad -x_3 - 2x_4$

$x_2 = a$

$x_3 = b$

$x_4 = c$

$$x = \begin{bmatrix} -2a - 2b - 4c \\ a \\ -b - 2c \\ b \\ c \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{N(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right)}$$

$A \to n$

~~0) Write the time you ended the exam and started to upload it.~~