MATH 511 - EXAM 3

1) For the given mapping is it a Linear Transform? If it is, what are the Kernel and Range?

a) The mapping from P_2 to P_3 such that $L(a\,x+b)=a\,b\,x^2+b\,x+a$

$L\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ab \\ b \\ a \end{bmatrix}$

$L(P_1+P_2) = \begin{bmatrix} (a+c)(b+d) \\ (b+d) \\ (a+c) \end{bmatrix}$

$L((ax+b)+(cx+d)) = (a+c)(b+d)x^2 + (b+d)x + (a+c)$

$\neq$ So... nope

$L(p_1)+L(p_2) = \begin{bmatrix} ab+cd \\ b+d \\ a+c \end{bmatrix}$

$L(ax+b) + L(cx+d) = (ab+cd)x^2 + (b+d)x + (a+c)$

b) The mapping from R^2 to R^3 such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$$

$\alpha_1 \begin{bmatrix} a \\ b \end{bmatrix} + \alpha_2 \begin{bmatrix} c \\ d \end{bmatrix}$

$L(\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} \alpha_1 a + \alpha_2 c + \alpha_1 b + \alpha_2 d \\ \alpha_1 b + \alpha_2 d \\ \alpha_1 a + \alpha_2 c \end{bmatrix}$

Yes

$= \alpha_1 \begin{bmatrix} a+b \\ b \\ a \end{bmatrix} + \alpha_2 \begin{bmatrix} c+d \\ d \\ c \end{bmatrix} = \alpha_1 L(V_1) + \alpha_2 L(V_2)$

$\mathrm{Ker}(L)$ is $\begin{bmatrix} a+b \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ So $a=0$ $b=0$

$\mathrm{Ker}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

$L\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \\ a \end{bmatrix}$

Range(L) is $a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$L\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

$\begin{bmatrix} x_1 \\ x_2 \\ x_1+2x_2 \end{bmatrix}$ $\quad \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

2) For the linear operator $L(\boldsymbol{x}) = [\, x_1, x_2, x_1 + 2x_2 \,]^T$ from $\mathbb{R}^2$ into $\mathbb{R}^3$ find the standard linear operator matrix, A.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \left[L\begin{pmatrix}1\\0\end{pmatrix} \; L\begin{pmatrix}0\\1\end{pmatrix} \right]$$

3) For the linear operator $L(\boldsymbol{p}) = x\boldsymbol{p}' + \boldsymbol{p}''$ on P_3 with standard basis $E = [(1), (x), (x^2)]$ and basis $B = [(1), (1 + x), (x + x^2)]$. Note: p' is the derivative of polynomial p.

$a + bx + cx^2$

B:

$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$

A

a) Find the matrix representation of L with respect to the standard basis, and call it A.

$$A = [\, \underline{L(P_1)} \;\; \underline{L(P_2)} \;\; \underline{L(P_3)} \,] = [\, 0, x, 2x^2 + 2 \,]$$

$$= \boxed{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

b) Write the matrix representation of L with respect to the non-standard basis B as a product of inverses, B, and/or A.

$$B^{-1} A B = \boxed{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} 1 & 1+x & x+x^2 \end{matrix} \end{array}}$$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ for [illegible] $e^x \to \begin{bmatrix}1\\0\end{bmatrix}$

4) Let V be the subspace of C[a,b] spanned by e^x, e^{-x} and let A_x be the anti-differentiation operator that also holds the constant of integration to be zero. Example: $A_x(2e^{-x}) = -2e^{-x}$. Find the matrix that represents $A_x()$ in the standard ordered basis e^x, e^{-x} and call that matrix A. Find the matrix that represents $A_x()$ in the non-standard ordered basis $\cosh(x)$, $\sinh(x)$ and call that matrix B. And write matrices A and B as $B = S^{-1}AS$ for some matrix S.

$A = [L(e_1)\ L(e_2)]$

$A = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

$A_x[e^x] = e^x \to \begin{bmatrix}1\\0\end{bmatrix}$

$A_x(e^{-x}) = -e^{-x} \to \begin{bmatrix}0\\-1\end{bmatrix}$

$\cosh(x) \to \begin{bmatrix}1/2\\1/2\end{bmatrix} \to \begin{bmatrix}1/2&1/2\\1/2&-1/2\end{bmatrix}$

$\sinh(x) \to \begin{bmatrix}1/2\\-1/2\end{bmatrix}$

$$\boxed{B = \begin{bmatrix}1/2&1/2\\1/2&-1/2\end{bmatrix}^{-1}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1/2&1/2\\1/2&-1/2\end{bmatrix}}$$

change of basis

or

$A_x(\cosh x) = \sinh x$

$A_x(\sinh x) = \cosh x$

$$\boxed{B = \begin{bmatrix}0&1\\1&0\end{bmatrix}}$$

5) For the pair of vectors $\boldsymbol{x} = (1, -2, 3)^T$ and $\boldsymbol{y} = (1, 1, 1)^T$, find the scalar projection $\boldsymbol{x}$ onto $\boldsymbol{y}$, the vector projection $\boldsymbol{p}$ of $\boldsymbol{x}$ onto $\boldsymbol{y}$, and verify that $(\boldsymbol{x} - \boldsymbol{p}) \perp \boldsymbol{p}$.

$$\alpha = \frac{x^T y}{\|y\|} = \boxed{\frac{2}{\sqrt{3}}} = \frac{x^T y}{(y^T y)^{1/2}}$$

$$p = \frac{x^T y}{y^T y}\, y = \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \boxed{\begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix}}$$

$$\boxed{x - p = \begin{bmatrix}1/3\\-8/3\\7/3\end{bmatrix}}$$

$$p^T(x - p) = \frac{2}{9} - \frac{16}{9} + \frac{14}{9} = 0$$

so $\perp$

6) For the pair of vectors $\boldsymbol{x} = (1, -2, 3)^T$ and $\boldsymbol{y} = (1, 1, 1)^T$, find the angle between the two vectors and the distance between the two vectors.

$$\cos\theta = \frac{x^T y}{\|x\|\|y\|} = \frac{2}{\sqrt{14}\sqrt{3}}$$

$$\boxed{\theta = \arccos\left(\frac{2}{\sqrt{42}}\right)}$$

$$\|x - y\| = \sqrt{0^2 + (3)^2 + (-2)^2} = \boxed{\sqrt{13}}$$

7) Let A be a 3x4 matrix. Considering A as a linear transform describe its domain and codomain.

a) For its codomain, is it possible to have the vector $(3, 1, 2)^T$ in the null space of A^T and $(-1, 0, 1)^T$ in the column space of A? Explain.

orthogonal

x $\quad$ y

$x^T y = -3 + 0 + 2 = -1 \neq 0$ so $\boxed{\text{No}}$ not orthogonal

x

b) For its domain, give an example of a vector in $R(A^T)$ if the vector $(1, -1, 1, -1)^T$ is in N(A).

need 1 vector $\perp$ to $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $\quad$ orthogonal.

Ex

$$\boxed{\text{so } y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$x^T y = 1 - 1 + 1 - 1 = 0$$

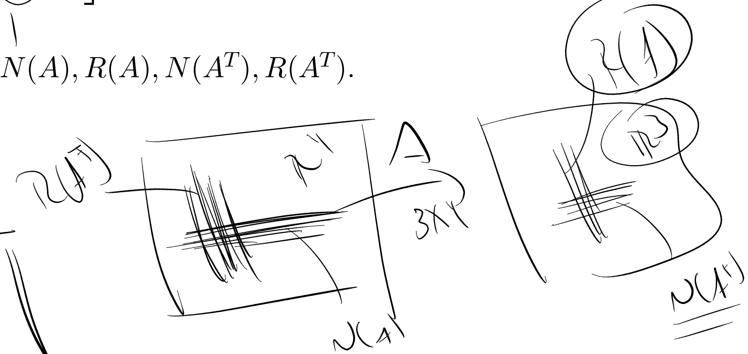
8) For the matrix …

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & -2 & 0 & -1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

Dim = 2

Find the basis for each of the fundamental subspaces $N(A), R(A), N(A^T), R(A^T)$.

$$\to U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



For $R(A) \to \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$R(A^T) \to \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$N(A) \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = \alpha \quad x_4 = \beta$

$x_3 = -\beta \quad x_1 = -2\alpha - \beta$

$$x = \begin{matrix} -2\alpha - \beta \\ \alpha \\ -\beta \\ \beta \end{matrix}$$

$N(A) \; \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$N(A^T) \to \underbrace{\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right]}_{A^T} \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = \alpha \quad x_2 = 0 \quad x_1 = -\alpha$

$N(A^T) \to \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

9) You have the following (x, y)-data points: $\{(-1,1), (0,1), (1,2), (2,2), (3,1)\}$. As was explained in the exam review ... setup the matrices and equation to solve the least-squares fit to the data by a polynomial $y = ax^2 + bx + c$. DO NOT solve it. Just get it to the point where you would only need to do the matrix arithmetic to solve it.

y

$$\begin{array}{ccc} x^0 & x^1 & x^2 \end{array} \left[\quad\right] \begin{bmatrix} c \\ b \\ a \end{bmatrix}$$

A

$$\begin{array}{ccc} x^2 & x^1 & x^0 \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

A

$$\longrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & 0 & 1 & 4 & 9 \\ -1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right)$$

10) Given vector space $\mathbb{R}^{2\times 3}$ verify the following function is an inner product ...

$\langle A, B\rangle = a_{11}b_{11} + 2a_{12}b_{12} + 3a_{13}b_{13} + 4a_{21}b_{21} + 5a_{22}b_{22} + 6a_{23}b_{23}$

① $\langle A, A\rangle = a_{11}^2 + 2a_{12}^2 + 3a_{13}^2 + 4a_{21}^2 + 5a_{22}^2 + 6a_{23}^2 \leftarrow$ all non-neg!

so if $\langle A, A\rangle = 0$ all $a_{ij} = 0$ ✓

② $\langle A, B\rangle$ by real number commutativity

$= b_{11}a_{11} + 2b_{12}a_{12} + \cdots + 6b_{23}a_{23} = \langle B, A\rangle$ ✓

③ $\langle \alpha A + \beta B, C\rangle = (\alpha a_{11} + \beta b_{11})c_{11} + 2(\alpha a_{12} + \beta b_{12})c_{12} + \cdots$

$\cdots + 6(\alpha a_{23} + \beta b_{23})c_{23}$

$= \alpha(a_{11}c_{11} + 2a_{12}c_{12} + \cdots + 6a_{23}c_{23})$

$+ \beta(b_{11}c_{11} + 2b_{12}c_{12} + \cdots + 6b_{23}c_{23})$

$= \alpha\langle A, C\rangle + \beta\langle B, C\rangle$ ✓

$\boxed{\text{true}}$

11) Given inner product space $\mathbb{R}^{2\times 3}$ with inner product ...

$$\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} = \sum_{i=1}^{3}\sum_{j=1}^{2} a_{ji}b_{ji}$$

find the projection of matrix C onto matrix D.

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$P = \frac{\langle C, D\rangle}{\langle D, D\rangle} D$$

$$= \frac{1+0-1+2+0-1}{1+0+1+1+1+1} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 1/5 & 0 & -1/5 \\ 1/5 & -1/5 & -1/5 \end{bmatrix}}$$