Math 511

"Eigen Stuff" (chapter 6)

Goal: Matrix $A v \overset{?}{=} \lambda v$ constant

A

→ becomes $(A - \lambda I) v = 0$ (non-trivial soln?)

→ becomes solve $\boxed{\det(A - \lambda I)} = 0$ charac. poly.

→ get λ_i

for each λ_i find $N(A - \lambda_i I)$ → eigen space

→ basis of space

→ eigen vectors

Properties

① $\lambda_1 + \lambda_2 + \dots + \lambda_k = a_{11} + a_{22} + \dots + a_{kk}$

call $a_{11} + a_{22} + \dots + a_{kk} = \text{trace}(A)$ ←

② $\lambda_1 \lambda_2 \cdots \lambda_k = \det(A)$

③ for any similar matrices $B = S^{-1} A S$

they have same eigen values

④ if λ_1 = complex number and its v_1 = complex number

then if $A = [a_{ij}]$ is all real numbers

→ then $\boxed{\text{complex conj.}}$ $\overline{\lambda_1}, \overline{v_1}$ are also eigen value / vectors

Complex why? $a+bi$ vs $a-bi$

(ex) Find $\boxed{\lambda_1 = 2+i}$ $v_1 = 0+i$

Know $\lambda_2 = 2-i$ $v_2 = 0-i$

Why?

(1) Study Markov process

$$v_1 = A v_0, \quad v_2 = A v_1, \quad v_3 = A v_2$$

$$\dots \quad \boxed{v_k = A^k v_0}$$

(ex) p. 50-52 turtle

$$L = \begin{bmatrix} 0 & 0 & 127 & 79 \\ 0.67 & 0.79 & 0 & 0 \\ 0 & .0006 & 0 & 0 \\ 0 & 0 & .81 & .8057 \end{bmatrix} \begin{array}{l} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \end{array} \begin{array}{l} <1\text{yr} \\ (1\text{-}21)\text{yr} \\ (22)\text{yr} \\ (23\text{-}54)\text{yr} \end{array}$$

$$t_1 = L\, t_0$$

$$t_k = L^k t_0$$

Why eigen values? eigen vectors?

eigen stuff

$A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

$\vdots$

$\rightarrow [A v_1 \; A v_2 \cdots A v_k] = [\lambda_1 v_1 \cdots \lambda_k v_k]$

$A[v_1 \cdots v_k] = [\lambda_1 v_1 \cdots \lambda_k v_k]$

let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_k \end{bmatrix}$ $\quad V = [v_1 \; v_2 \cdots v_k]$

$$\boxed{AV = VD}$$

if v_i are <u>ind.</u> $\rightarrow V$ is invertable

$$\boxed{A = V D V^{-1}}$$

so A, D are similar

so $A^k = A \cdot A \cdot A \cdot A \cdots A$

$= (VD\underbrace{V^{-1})(V}D\underbrace{V^{-1})\cdots(V}DV^{-1})$

$= V D \cdot D \cdots D V^{-1} = V D^k V^{-1}$

but $D^k = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_k \end{pmatrix}^{k} = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_k \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_k \end{pmatrix} \cdots$

$D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ 0 & & \lambda_k^k \end{pmatrix}$ different <u>k's</u>

So back to Markov process

turtle $t_n = L^n t_0$

if $L = V D V^{-1} \to L^n = V D^n V^{-1}$

V eigenvectors, D eigenvalues

$(\lambda_i < 1)^n \to 0$

$(\lambda_i = 1)^n \to 1$

turtles @ year-n

$$\underbrace{t_n}_{[t_n]_{\text{standard}}} = V \begin{pmatrix} \boxed{\lambda_1^n} & & 0 \\ & \boxed{\lambda_2^n} & \\ 0 & & \boxed{\lambda_k^n} \end{pmatrix} \underbrace{V^{-1} [t_0]_{\text{standard}}}_{[t_0]_{\text{eigenvector basis}}}$$

p. 50-52 turtle

$$L = \begin{bmatrix} 0 & 0 & 127 & 79 \\ 0.67 & 0.79 & 0 & 0 \\ 0 & .0006 & 0 & 0 \\ 0 & 0 & .81 & .8087 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$

t_1 — <1 yr, t_2 — (1-21) yr, t_3 — (22) yr, t_4 — (23-54) yr

Heart of application: Diagonalization

$AV = VD$ (for ind vecs)

$A = VDV^{-1}$

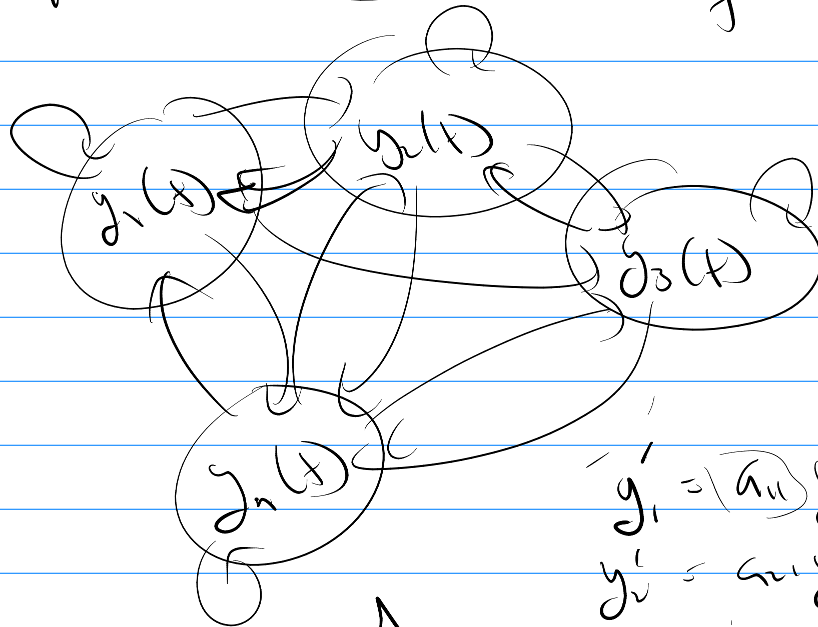
or $D = V^{-1}AV$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

$V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_k]$

only 1st deriv

Systems of 1st order diff eq'ns



$y_i(t)$ = amount of "stuff" in y_i @ time t.

$$y_i' = \frac{dy_i}{dt}$$

$$y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n$$
$$y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n$$

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

Call $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $Y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$

$$Y' = AY$$

diff eq'ns #1 tech is guess & check $y' = kae^{at}$

1D: $y' = ay$ $y = ke^{at}$

Guess

$$y_i = e^{\lambda_i t} V_i$$

$$Y' = A Y$$

do eigen stuff λ_i, V_i

$$Y = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + \dots + c_k e^{\lambda_k t} V_k$$

→ find c_i by initial value

or if $A = VDV^{-1}$

$$Y = V \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_k t} \end{bmatrix} V^{-1} [Y_0]$$ initial values