Math 511

C.2 (exam type probs)

① given system

$y_1' = 2y_1 + 0y_2 - 6y_3$
$y_2' = 1y_1 + 0y_2 - 3y_3$
$y_3' = 0y_1 + 1y_2 - 2y_3$

$y_1(0) = 2$, $y_2(0) = 2$, $y_3(0) = 2$

$$Y_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$Y' = \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} Y \quad , \quad Y_0 \text{ initial values}$$

Ans: if given $A \rightarrow$ eigen values are $\lambda_1, \lambda_2, \lambda_3$
eigen vectors are V_1, V_2, V_3

$$Y = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2 + C_3 e^{\lambda_3 t} V_3$$

find C_1, C_2, C_3 by using Y_0 initial values

So

$$A = \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \lambda_i \,?\; V_i \,? \sim AV = \lambda V$$

$(A - \lambda I)V = 0$

#1

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{bmatrix}$$

Show work

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)\begin{vmatrix} -\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & -6 \\ 1 & -2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[-\lambda(-2-\lambda)+3\right] - 6$$

$$= (2-\lambda)\left[\lambda^2+2\lambda+3\right] - 6$$

$$= (-\lambda^3+\lambda+6) - 6 = -\lambda^3+\lambda$$

$$= -\lambda(\lambda^2-1)$$

$$= -\lambda(\lambda+1)(\lambda-1)$$

set $= 0$

$$\lambda_1 = 0 \qquad \lambda_2 = -1 \qquad \lambda_3 = 1$$

$$A-\lambda I = \begin{bmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{bmatrix}$$

$\lambda_1 = 0 \rightsquigarrow N\begin{pmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}$

$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 0 & -6 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free!

$x_1 = 3\alpha$

$x_2 = 2\alpha$

$x_3 = \alpha$

$$x = \alpha\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1 \quad N\begin{pmatrix} 3 & 0 & -6 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{pmatrix}$

$v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 3 & 0 & -6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x = \begin{bmatrix} 2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

x_3 free

$\lambda_3 = 1 \Rightarrow N\begin{pmatrix} 1 & 0 & -6 \\ 1 & -1 & -3 \\ 0 & 1 & -3 \end{pmatrix}$ $\left[\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 1 & -1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$

$V_3 = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x_3 free

$X = \begin{bmatrix} 6\lambda \\ 3\lambda \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_1 = 0$ $\quad V_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\quad V_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_3 = 1$ $\quad V_3 = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

Soln $Y = c_1 e^{0t}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

$Y = c_1\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

Initial Value

if you know @ $t = 0$ $\quad Y_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ you can find c_1, c_2, c_3

$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

"odd" example?

(ex) $\lambda_1 = -3$ $\rightsquigarrow$ $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\mathbb{X} = \begin{bmatrix} \alpha \\ -2\beta \\ \beta \end{bmatrix}$

$x_1 = \alpha$ $x_3 = \beta$

So $N(\ ) = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ eigenspace

$\lambda_1 = -3$ $\mathbb{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\mathbb{V}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

(ex) $\lambda_3 = -1$ $\mathbb{V}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$AV = VD$

$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

vs

$\begin{bmatrix} 0 & 1 & 2 \\ -2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$