Math 415

Q's

8. Suppose a is an integer. If $5 \mid 2a$, then $5 \mid a$.
9. Suppose a is an integer. If $7 \mid 4a$, then $7 \mid a$.

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Proof. Suppose $7 \mid 4a$.
By definition of divisibility, this means $\boxed{4a = 7c}$ for some integer c.
Since $4a = 2(2a)$ it follows that $4a$ is even, and since $4a = 7c$, we know $\boxed{7c \text{ is even}}$.
But then c can't be odd, because that would make $7c$ odd, not even.
Thus c is even, so $c = 2d$ for some integer d.
Now go back to the equation $4a = 7c$ and plug in $c = 2d$. We get $4a = 14d$.
Dividing both sides by 2 gives $\boxed{2a = 7d}$.
Now, since $2a = 7d$, it follows that $7d$ is even, and thus d cannot be odd.
Then d is even, so $d = 2e$ for some integer e.
Plugging $d = 2e$ back into $2a = 7d$ gives $2a = 14e$.
Dividing both sides of $2a = 14e$ by 2 produces $a = 7e$.
Finally, the equation $\boxed{a = 7e \text{ means that } 7 \mid a}$, by definition of divisibility. ■

goal.

?

Sketch

try direct

assume $5 \mid 2a \rightarrow$ show $5 \mid a$

Know: $m \mid n$ means $m \cdot k = n$ $k \in \mathbb{Z}$

"n has a factor of m"
"n is a multiple of m"

assume $5 \mid 2a$, then $5 \cdot k = 2a$, $k \in \mathbb{Z}$

Because $2a$ is an even number, $5k$ is even, with 5 being odd, this gives k is even. (Why? B/c odd $\cdot$ odd = odd and odd $\cdot$ even = even)

Lemma The product of odds is odd.

Pf Let $m = 2k_1+1$ and $n = 2k_2+1$ $k_1, k_2 \in \mathbb{Z}$ be any two odd integers. Consider mn,

$mn = (2k_1+1)(2k_2+1) = (4k_1k_2 + 2k_1 + 2k_2) + 1$

Continued

Lemma The product of odds is odd.

pf Let $m = 2k_1+1$ and $n = 2k_2+1$ $k_1, k_2 \in \mathbb{Z}$ be any two odd integers. Consider mn,

$$mn = (2k_1+1)(2k_2+1) = (4k_1k_2 + 2k_1 + 2k_2) + 1$$

$$= 2(2k_1k_2 + k_1 + k_2) + 1$$

by closure properties of mult. and add of integers

$2k_1k_2 + k_1 + k_2 \in \mathbb{Z}$ call it k_3

gives $mn = 2k_3 + 1$ an odd.

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back to proof of $5|2a \to 5|a$

assumed $5|2a$ or $5k = 2a$ for $k \in \mathbb{Z}$

by lemma (odd · odd = odd) gives

k must be even.

(Maybe prove odd · even = even as a lemma as well)

So let $k = 2l$, $l \in \mathbb{Z}$.

and $5k = 2a$ becomes $5 \cdot \cancel{2}l = \cancel{2}a$

and $5l = a$ or $5|a$.

Title:

Author: You

abstract

Start We will study the properties of mult. by parity.

lemma 1 odd · odd

lemma 2 even · even

lemma 3 odd · even (Note: by commutativity odd · even is same as even · odd)

thm $5 \mid 2a \rightarrow 5 \mid a$

Direct: assume hyp and show conclusion

Contrapositive: $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$

show by direct proof

Lemma if a^2 is even, then a is even.

pf: Using the contrapositive,

if a is odd, then a^2 is odd.

proving this directly gives assume $a = 2k+1$, $k \in \mathbb{Z}$.

then $a^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$

by closure properties $2k^2+2k \in \mathbb{Z}$, call it k_2

$a^2 = 2k_2+1$ an odd

∎

Statement $\sqrt{2}$ is irrational.

Scratch: ① $\sqrt{2}$??? $\sqrt{4} = 2$

$x^2-4=0$

$(x+2)(x-2)=0$

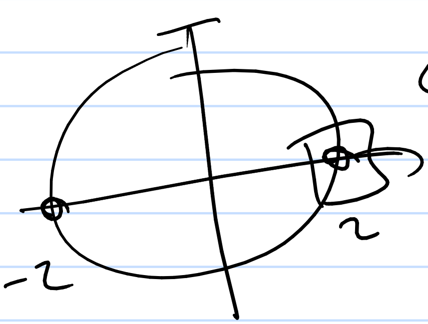
$x=-2$ $x=2$

$x^2-4=0$

$x^2=4$

$x = \pm\boxed{\sqrt{4}}^{+}$

$x = \pm 2$



$\mathbb{C}$

$\sqrt[3]{-8}$

$\sqrt{4}$

$\sqrt{x} = 4$

$\boxed{n \circ n \overset{?}{=} x}$

① So $\sqrt{2}$ means $\sqrt{2} = n$

find n so that $n \cdot n = 2$

or $n^2 = 2$

② not irrational

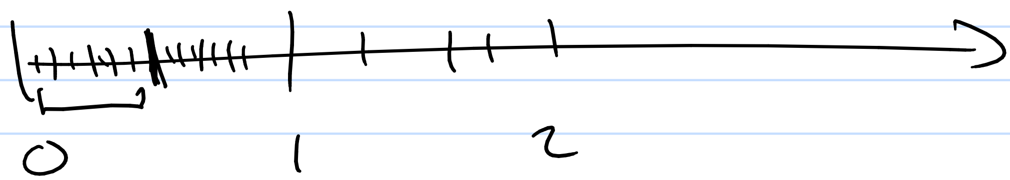
rational?

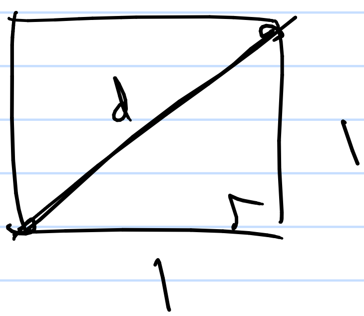
Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rationals: $\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \wedge b \neq 0$ $\wedge$ they have **no** common factors $\}$

$\frac{7}{9} = 7\left(\frac{1}{9}\right)$

$\boxed{\text{ruler:}}$





$1^2 + 1^2 = d^2$

$d^2 = 2$

$d \cdot d = 2$

$d = \sqrt{2} = \left(\frac{a}{b}\right)$

$\frac{a}{b} = \sqrt{2}$

$\frac{a^2}{b^2} = 2$ b/c $b \neq 0$

$a^2 = 2b^2$

so a^2 is even. by lemma a is even

let $a = 2k$ $k \in \mathbb{Z}$

$\Rightarrow (2k)^2 = 2b^2 \rightarrow \overset{2}{\cancel{4}}k^2 = \cancel{2}b^2$

$\rightarrow 2k^2 = b^2 \rightarrow b^2$ is even $\rightarrow$ b is even

Proof By Contradiction

goal: $(\text{Statement}) \equiv T$

or rather show: $\boxed{\neg(\text{Statement}) \equiv F}$ Proof by Contradiction

typical statements: ① Conclusion

show

show $(\neg \text{Conclusion}) \equiv F$

② $p \to q$

show

$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv (p \wedge \neg q) \equiv F$

Heart: $\boxed{\neg C \wedge C \equiv F}$

In some statement C you find $C \wedge \neg C$ in the argument.

Fact: $\sqrt{2}$ is irrational

pf We will use a proof by contradiction. So, assume $\sqrt{2}$ is rational. By definition

$\sqrt{2} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$, (a, b have no common factors.)

Now $2 = a^2/b^2$ given $2b^2 = a^2$ b/c $b \neq 0$
From this we see a^2 is even. By lemma a is also even
Let $a = 2k$, $k \in \mathbb{Z}$ give $2b^2 = 4k^2$ or $b^2 = 2k^2$
From this we see b^2 is even. By lemma b is also even
therefore (a, b have no common factors) $\wedge$ (a, b have a common factor of 2) is always False.

So $\sqrt{2}$ is rational $\equiv F$ false

$\sqrt{2}$ is irrational is true

QED

Q) prove $\sqrt{3}$ is irrational

pf) try contradiction.

So assume $\sqrt{3}$ is rational. $\sqrt{3} = \frac{a}{b}$ with facts

$3 = \frac{a^2}{b^2} \to 3b^2 = a^2 \to ???$

lemma! $\boxed{3 \mid a^2 \to 3 \mid a}$ do @ home.

$\equiv 3 \nmid a \to 3 \nmid a^2$

Background $3 \nmid a$ means $\boxed{\begin{array}{l} a = 3k+1 \\ \quad \text{or} \\ a = 3k+2 \end{array}}$

Building to fundamental Thm of Arithmetic

all integers are primes or a unique product of primes written in non-dec. order.

① Integers are either prime or Composite

excluding.   not prime.

② A prime is an integer whose factors are only 1 and itself.

③ 1 is not prime and not composite

1	2	3	4	5	6	7	8	--
	prime	prime	composite	prime	composite			
			$4=2^2$		$6=2\cdot 3$			

Conjecture: there are infinitely many primes.

pf.