Math 415 Ch 7 (1, 2, 6, 9, 12, 17) Due Next Monday
(Book of Proof)

Q's Ch 5 (18)

17. If n is odd, then $8 \mid (n^2-1)$.
18. If $a,b \in \mathbb{Z}$, then $(a+b)^3 \equiv a^3+b^3 \pmod 3$.
19. Let $a,b,c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod n$ and $a \equiv c \pmod n$, then $c \equiv b \pmod n$.
20. If $a \in \mathbb{Z}$ and $a \equiv 1 \pmod 5$, then $a^2 \equiv 1 \pmod 5$.
21. Let $a,b \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \pmod n$, then $a^3 \equiv b^3 \pmod n$.

(1) $a \mid b$ means $a \cdot k = b$, $k \in \mathbb{Z}$ (divide)

(2) $a \equiv b \pmod n$ or $a \underset{n}{\equiv} b$ (congruent)

$a, b, n \in \mathbb{Z}$ means (a) $n \mid (b-a)$

(b) $a = b + k \cdot n$, $k \in \mathbb{Z}$

(c) $a \bmod n = b \bmod n$

18 $a,b \in \mathbb{Z} \to (a+b)^3 \underset{3}{\equiv} a^3+b^3$

Pf By a direct proof assume $a,b \in \mathbb{Z}$, then

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

Scratch

3 is a factor of $3a^2b+3ab^2$

$3 \mid 3a^2b+3ab^2$

$3 \mid \cancel{a^3}+3a^2b+3ab^2+\cancel{b^3}-\cancel{a^3}+b^3$

$3 \mid (a+b)^3-(a^3+b^3)$

Goal?!

$\boxed{(a+b)^3 \underset{3}{\equiv} (a^3+b^3)}$

Pf Assume $a,b \in \mathbb{Z}$.

Now $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

therefore: $(a+b)^3 - (a^3+b^3) = 3a^2b + 3ab^2$

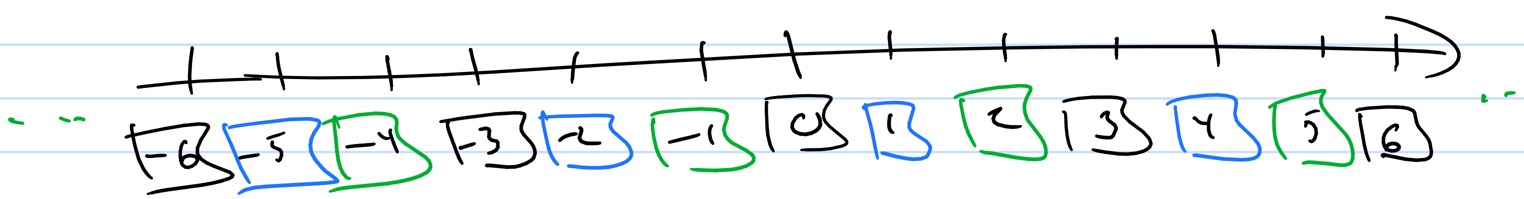
$$= 3(a^2b + ab^2)$$

b/c $3 \mid 3(a^2b+ab^2)$ gives $3 \mid (a+b)^3 - (a^3+b^3)$

by Def. $(a+b)^3 \underset{3}{\equiv} (a^3+b^3)$ ∎

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \underset{3}{\equiv} a^3 + 0 + 0 + b^3 = a^3+b^3$

Now $3 \underset{3}{\equiv} 0$



ex $(a,b,c)_{10}$ ex $(1,2,3)_{10} = 1\cdot 10^2 + 2\cdot 10^1 + 3$

mod 3 $\qquad 10 \underset{3}{\equiv} 1$ $\qquad \underset{3}{\equiv} 1+2+3$

Q

8. Suppose $x \in \mathbb{R}$. If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \ge 0$, then $x \ge 0$.

Pf We will prove the contrapositive statement:

if $x < 0$, then $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$

which is if x is negative then $+x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4$ is negative.

So assume x is negative. Then x^5 is negative, $-4x^4$ is negative,

or assume $x < 0$.

$x \cdot x > 0 \cdot x$ $\qquad$ $x^2 > 0$

$x^2 \cdot x < 0 \cdot x$ $\qquad$ $x^3 < 0$

Finish

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assume $x < 0$. Consider the table

$x < 0$				(1)
$x \cdot x > 0 \cdot x$	so	$x^2 > 0$		(2)
$x^2 \cdot x < 0 \cdot x$	so	$x^3 < 0$		(3)
$x^3 \cdot x > 0 \cdot x$	so	$x^4 > 0$		(4)
$x^4 \cdot x < 0 \cdot x$	so	$x^5 < 0$		(5)

Now $x^5 < 0$ and $-4x^4 > 0$ by (4) $\qquad$ $x^4 > 0$

$-4x^4 < 0 \cdot (-4)$

$-4x^4 < 0$

All these are $p \to q$ in some format

① $p \to q$

② $\neg q \to \neg p$

③ show $(p \wedge \neg q) \equiv F$

④ $(p_1 \leftrightarrow p_2) \equiv (p_1 \to p_2) \wedge (p_2 \to p_1)$

⑤ $p_1 \equiv p_2 \equiv p_3 \equiv \cdots \equiv p_k$

show $(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_k \to p_1)$

⑥ $(p_1 \vee p_2 \vee \cdots \vee p_k) \to q$

$\equiv (p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_k \to q)$

(ex) Show there are no integer solutions to $x^2 + 3y^2 = 8$

Scratch

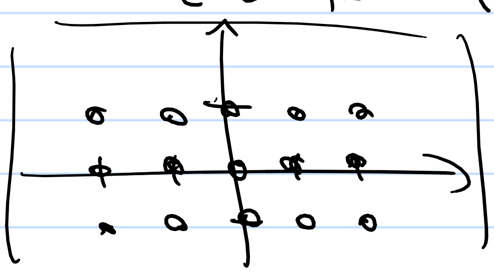
① consider $x^2 + y^2 = z^2$ — $(3,4,5)$ is an integer solution (x y z)

② consider $3x + y = 4$ — $(1,1)$ is an int. solution

$(2,-2)$ is an int. solution

Pf

well possible integer solutions are hold one variable to be zero then $|x| \le 2$, $|y| \le 1$ giving the following grid



for all possible integer solutions. And without loss of generality we can consider only $(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)$ because the powers are both 2.

Case 1: $x=0, y=0$ and $0^2 + 3(0)^2 \neq 8$

⋮

Other Conjectures

(1) Existence. $\exists x\, P(x)$

(A) Constructive. Find the $\boxed{x \text{ that has the property.}}$ ← witness

(ex) there are two successive integers where one is a square and one is a cube.

Pf: $\boxed{1}$, 2, 3, $\boxed{4}$, 5, 6, 7, $\boxed{8, \boxed{9}}$, 10, 11, 12, 13, 14, 15, $\boxed{16}$, 17, 18, 19, 20, 21, 22, 23, 24, $\boxed{25}$, 26, 27, 28, 29, ...

Square $\square$
cube $\bigcirc$

$2^3 = 8$, $9 = 3^2$ are successive ints

(ex) there is a real number, r, such that $a \cdot r + b = 0$ if $a \neq 0$

Pf: $r = -\frac{b}{a}$ b/c $a\left(-\frac{b}{a}\right) + b = 0$

∎

(B) Non-Constructive. Don't find witness, but show it exists.

Ex: there is an irrational to an irrational power that is rational.

Pf: consider the only irrational we have proved so far: $\sqrt{2}$.

Let $r = \sqrt{2}^{\sqrt{2}}$. It is a real so it is rational or irrational.

$\boxed{\text{Case 1}}$ $\sqrt{2}^{\sqrt{2}}$ is rational. And therefore it is my witness.

or

$\boxed{\text{Case 2}}$ $\sqrt{2}^{\sqrt{2}}$ is irrational. So consider it $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$ which gives $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ as my witness.

<u>Uniqueness</u>