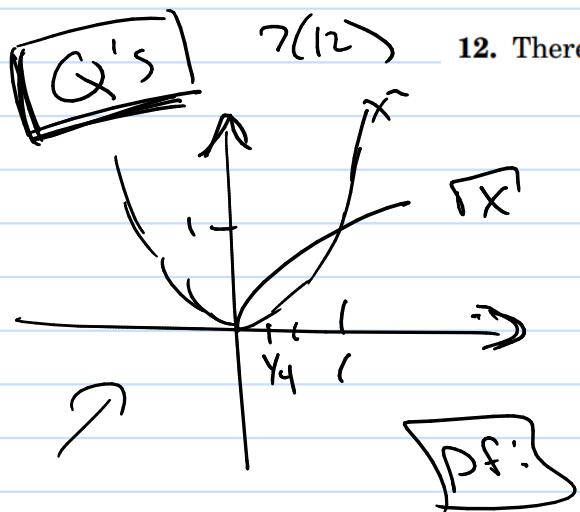


Math 415



12. There exists a positive real number x for which $x^2 < \sqrt{x}$.

$$x = \sqrt{4} \quad \hat{x} \Rightarrow (\sqrt{y_{16}})^2 = \frac{1}{16}$$

$$\sqrt{x} \Rightarrow \sqrt{\sqrt{y_{16}}} = \frac{1}{2}$$

$$\frac{1}{16} < \frac{1}{2}$$

Consider $x = \sqrt{4}$, $f(x) = \hat{x}$, and $g(x) = \sqrt{x}$.

$$f(\sqrt{4}) = \sqrt{16}, \quad g(\sqrt{4}) = \sqrt{2}$$

We see that for $x = \sqrt{4}$ $\hat{x} < \sqrt{x}$.

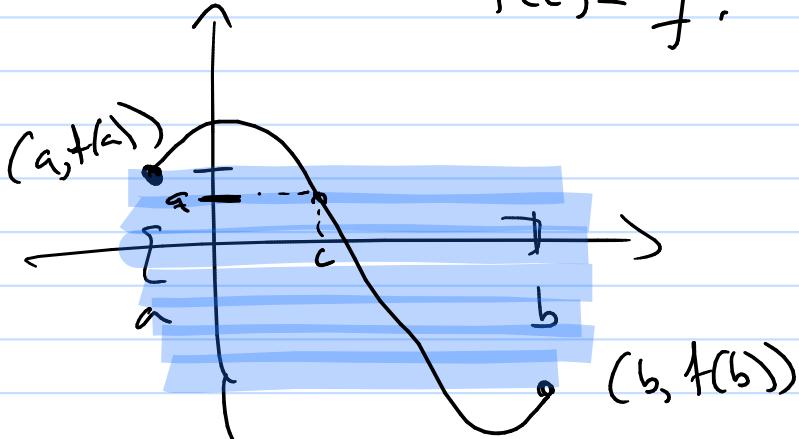


Intermediate Value Theorem for Calculus

If $f(x)$ is continuous on a closed interval $[a, b]$,

then for any real between $f(a)$ and $f(b)$, call it \hat{f} , there exists some c between a and b such that

$$f(c) = \hat{f}.$$



Conjecture: there is a pos. real root for $f(x) = x^2 - 2$

(13)

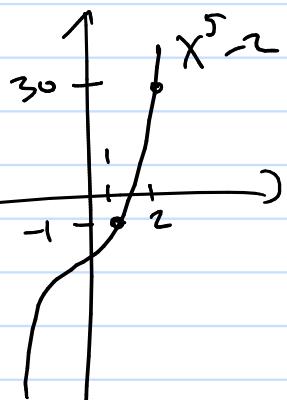
Conjecture: there is a pos. real root for $f(x) = x^2 - 4$

Let $x = 2$, $f(2) = 2^2 - 4 = 0$
 So $x = 2$ is a pos. real root.

Conjecture: there is a pos. real root for $f(x) = x^5 - 2$

TPF

Consider $x=1$, $f(1)=-1$ and $x=2$, $f(2)=30$



by I.V.T. for any number between -1 and 30 there is $x=c$ such that

$f(c)$ = that value between -1 and 30.

Therefore (by Univ. Instatka) there

is some $x=c$ such that $f(c)=0$.



Construction:

Conjecture: For rational numbers p, q and $p \neq q$ there exists a rational number, r , such that r is between p and q .

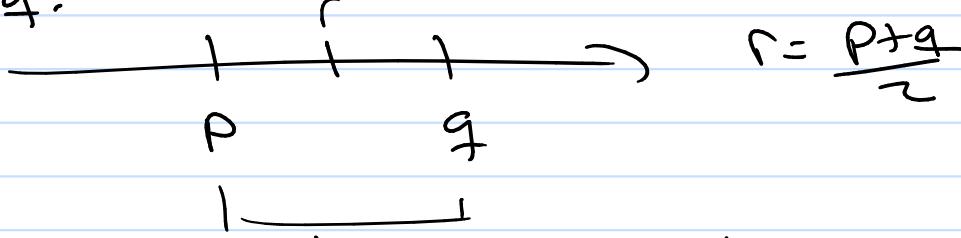
TPF

Consider the average of p, q . Let $r = \frac{p+q}{2}$

now r is between p, q and by closure properties of addition and multiplication of reals

$$r = \frac{p+q}{2} = \left(\frac{1}{2}\right)(p+q) \text{ is rational.}$$

Ques: is r between p, q ? What loss of generality
let $p < q$.



$$\text{distance from } r \text{ to } p \text{ is } r-p = \frac{1}{2}p + \frac{1}{2}q - p = \frac{1}{2}q - \frac{1}{2}p = \frac{1}{2}d$$

Q

(Biconditional)?

11. Suppose $a, b \in \mathbb{Z}$. Prove that $(a-3)b^2$ is even if and only if a is odd or b is even.
12. There exists a positive real number x for which $x^2 < \sqrt{x}$.
13. Suppose $a, b \in \mathbb{Z}$. If $a+b$ is odd, then $a^2 + b^2$ is odd.
14. Suppose $a \in \mathbb{Z}$. Then $a^2 | a$ if and only if $a \in \{-1, 0, 1\}$.

11 $\left[(a-3)b^2 \text{ is even} \right] \text{ iff } \left[a \text{ is odd or } b \text{ is even} \right]$

[PB] case 1 $\left[(a-3)b^2 \text{ is even} \right] \rightarrow \left[a \text{ is odd or } b \text{ is even} \right]$

by contraposition we will prove,

$$\left[a \text{ is even and } b \text{ is odd} \right] \rightarrow \left[\underline{(a-3)b^2} \text{ is odd} \right]$$

assume a is even and b is odd, therefore

$$\left[\underline{a = 2k_1}, k_1 \in \mathbb{Z} \right] \text{ and } \left[\underline{b = 2k_2 + 1}, k_2 \in \mathbb{Z} \right]$$

$$\text{this gives } (a-3)b^2 = (2k_1 - 3)(2k_2 + 1)^2$$

$$= (2k_1 - 3)(4k_2^2 + 4k_2 + 1)$$

$$= 8k_1k_2^2 + 8k_1k_2 + 2k_1 - 12k_2^2 - 12k_2 - 3$$

$$= [8k_1k_2^2 + 8k_1k_2 + 2k_1 - 12k_2^2 - 12k_2 - 4] + 1$$

$$= \dots = 2[\text{integer}] + 1 \text{ is an odd.}$$

case 2 $\left[a \text{ is odd or } b \text{ is even} \right] \rightarrow \left[(a-3)b^2 \text{ is even} \right]$

case 2.1 $\left[a \text{ is odd} \right] \rightarrow \left[(a-3)b^2 \text{ is even} \right]$

and

case 2.2 $\left[b \text{ is even} \right] \rightarrow \left[(a-3)b^2 \text{ is even} \right]$

(Finish Case two)



Uniqueness

$a, b, c \in \mathbb{R}$

Conjecture

for $a \neq 0$, there is a unique real soln
to $ax + b = c$

(Pf)

To prove existence, consider $x = \frac{-b}{a}$. This is
a real number because $a \neq 0$ and

$$a\left(\frac{-b}{a}\right) + b = -b + b = c$$

so it is a solution.

To prove uniqueness assume we have some other
solution, call it y . So, $ay + b = c$ as
well. But, $ax + b = c$ gives $ax + b = ay + b$

and $ax = ay$, therefore $x = y$

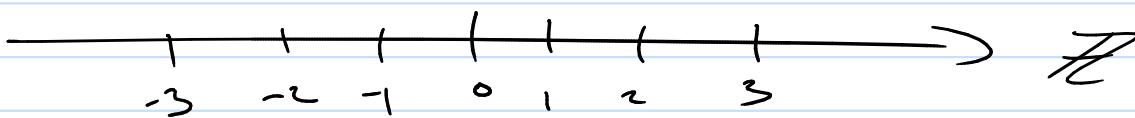


Next Big Proof Type: Induction

Ch 10 f Book of Proof

Given a well ordered set of elements.

Elements are $\{e_1, e_2, e_3, e_4, \dots\}$



for well ordered elements the following is a tautology.

$$\left[P(\text{1st element}) \wedge \forall k \{ (P(1) \wedge P(2) \wedge \dots \wedge P(k)) \rightarrow P(k+1) \} \right] \rightarrow \forall n P(n)$$

or in e_i notation

$$\left[P(e_1) \wedge \forall k \{ (P(e_1) \wedge P(e_2) \wedge \dots \wedge P(e_k)) \rightarrow P(e_{k+1}) \} \right] \rightarrow \forall n P(e_n)$$

Show this True

want T?

Showing the left side is to show...

① $P(e_1)$ is true? Basis Step

② $\{P(e_1) \wedge P(e_2) \wedge \dots \wedge P(e_k)\} \rightarrow P(e_{k+1})$ is true?

Inductive Step

Note:

$$\boxed{(P_1 \wedge P_2 \wedge P_3) \rightarrow q}$$

$$= (P_1 \rightarrow q) \vee (P_2 \rightarrow q) \vee (P_3 \rightarrow q)$$

$$= (P_1 \wedge P_2) \rightarrow q \times (P_3 \rightarrow q)$$

$$\{P(e_1) \wedge P(e_2) \wedge \dots \wedge P(e_k)\} \rightarrow P(e_{k+1})$$

$$= \underbrace{\{P(e_1) \rightarrow P(e_{k+1})\}}_{\text{---}} \times \dots \times \underbrace{\{P(e_f) \rightarrow P(e_{k+1})\}}_{\text{---}}$$

Ex) If $n \in \mathbb{N}$, then $1 + z + z^2 + z^3 + z^4 + \dots + z^n = z^{n+1} - 1$

IPB Direct

Assume $n \in \mathbb{N}$, so n is finite.

Consider

$$1 + z + z^2 + \dots + z^n = S$$

then $zS = z(1 + z + z^2 + \dots + z^n)$
 $zS = z + z^2 + z^3 + \dots + z^{n+1}$

This gives $zS - S = (z + z^2 + \dots + z^{n+1}) - (1 + z + \dots + z^n)$

$$S = z^{n+1} - 1$$

so $1 + z + z^2 + \dots + z^n = z^{n+1} - 1$

IPB

IPB (by induction)

Case 1 ($n=0$)

$$z^0 \stackrel{?}{=} z^{0+1} - 1$$

Basis? (Show this)

Case 2 ($n=1$)

$$z^0 + z^1 \stackrel{?}{=} z^{1+1} - 1$$

Case 3 ($n=2$)

$$z^0 + z^1 + z^2 \stackrel{?}{=} z^{2+1} - 1$$

Case 4 ($n=3$)

$$z^0 + z^1 + z^2 + z^3 \stackrel{?}{=} z^{3+1} - 1$$

⋮

($n=k$)

$$1 + z + z^2 + \dots + z^k \stackrel{?}{=} z^{k+1} - 1$$

($n=k+1$) $1 + z + z^2 + \dots + z^{k+1} \stackrel{?}{=} z^{k+2} - 1$

Inductive?
Assume
these
and
show