Math 415

Book of Proof

Ch 10 Induction 10(1, 2, 3, 4, 8, 12, 16, 25, 26, 30)

Due Next Wed.

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for well ordered sets of $e_1, e_2, e_3, e_4, \ldots$

for propositional function $P(x)$: "x has $\boxed{\text{predicate } P}$"

testable property

want to say all elements of U.D. have $P(x)$ is true.

$$\boxed{\forall n \; P(e_n) \text{ is true}}$$

up to now we did finite cases.. — ∞ cases!

ex. all numbers that are not divisible by 3 have ... (Predicate)

$\underbrace{1, 2, 4, 5, 7, 8, 10, 11 \ldots}$ $\boxed{\text{is } \infty}$

but they are really $\boxed{\begin{array}{c} 3k+1 \\ \text{case 1} \end{array}}$ or $\boxed{\begin{array}{c} 3k+2 \\ \text{case 2} \end{array}}$ $k \in \mathbb{Z}$

What about $\forall n \; P(n)$ where we can not find finite cases?!

Induction works only for a well ordered set of elements..

$e_1, e_2, e_3, e_4, \ldots$

For a well ordered set $e_1, e_2, e_3, \ldots$ the following is always true.

$$\underbrace{\{P(e_1) \wedge \forall k\,[(P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)) \rightarrow P(e_{k+1})]\}}\rightarrow \underbrace{\forall n\, P(e_n)}$$

Proof of $\forall n\, P(e_n)$ by induction is to show

① $\boxed{\text{Basis Step}}$ Show $P(e_1)$ is true

② $\boxed{\text{Inductive Step}}$

Inductive hypothesis

by Direct Proof

Assume $\boxed{P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)}$ are true

Show $P(e_{k+1})$ is true.

Note: (a) if when you show $P(e_{k+1})$ is true and you only used the truth of $P(e_k)$

call this $\boxed{\text{weak induction}}$

(b) if you need cases besides $P(e_k)$ call it $\boxed{\text{strong induction}}$

contradiction (smallest counter example)

Assume $\underbrace{P(e_1) \wedge P(e_2) \wedge \cdots \wedge P(e_k)}_{\text{all true}} \wedge \underbrace{P(e_{k+1})}_{\text{is false}}$

e_{k+1} is the smallest counter example.

show we get some $q \wedge \neg q \equiv F$

example $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

Gauss' Proof: let $S = 1+2+3+\cdots+n$

$S = n+(n-1)+(n-2)+\cdots+1$

gives $2S = (n+1)+(n+1)+\cdots+(n+1) = n(n+1)$

gives $S = \dfrac{n(n+1)}{2}$

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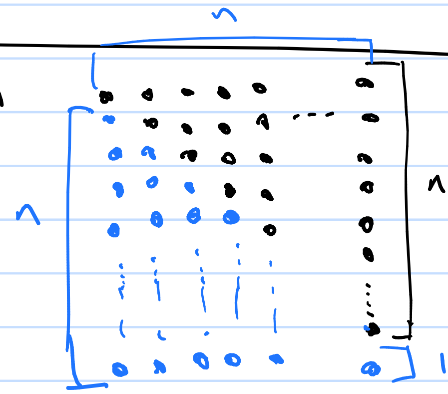
graph proof: $1+2+3+4+\cdots+n$

$n+(n-1)+(n-2)+\cdots+1$

Area $= n(n+1)$

so $\frac{1}{2}$ Area $= \boxed{\dfrac{n(n+1)}{2}}$



Induction? "$1+2+\cdots+n = \dfrac{n(n+1)}{2}$" : $P(n)$

e_1 : $(n=1)$ $P(1)$: "$1 = \dfrac{1(1+1)}{2}$"

e_2 : $(n=2)$ $P(2)$: "$1+2 = \dfrac{2(2+1)}{2}$"

e_3 : $(n=3)$ $P(3)$: "$1+2+3 = \dfrac{3(3+1)}{2}$"

e_4 : $(n=4)$ $P(4)$: "$1+2+3+4 = \dfrac{4(4+1)}{2}$"

$\vdots$

e_k : $(n=k)$ $P(k)$: "$1+2+3+4+\cdots+k = \dfrac{k(k+1)}{2}$"

e_{k+1} : $(n=k+1)$ $P(k+1)$: "$1+2+3+\cdots+k+(k+1) = \dfrac{(k+1)(k+1+1)}{2}$"

$\vdots$

Scratch

Basis Step show "$1 = \dfrac{1(1+1)}{2}$" is true (it is!)

Inductive Step Assume $P(e_1)\wedge P(e_2)\wedge\cdots\wedge P(e_k)$ are all true.

$P(e_k)$ so $\boxed{1+2+4+\cdots+k = \dfrac{k(k+1)}{2}}$ is true

$$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + \frac{(k+1)}{1}$$

$$1+2+\cdots+(k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$1+2+\cdots+(k+1) = \frac{k(k+1)+2(k+1)}{2}$$

$P(k+1)$ $$1+2+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$$

(IS) Inductive Step — we need an equality to be true..

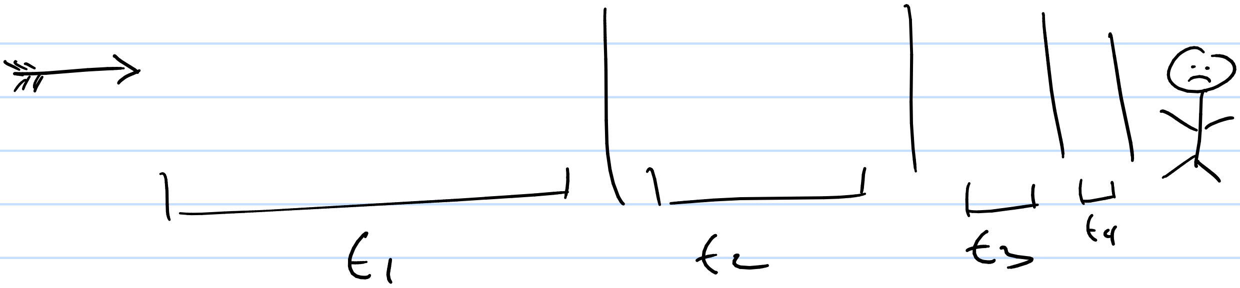
what?: $$1+2+\cdots+k+(k+1) \overset{?}{=} \frac{(k+1)(k+2)}{2}$$

$$1+2+3+\cdots+k+(k+1)$$

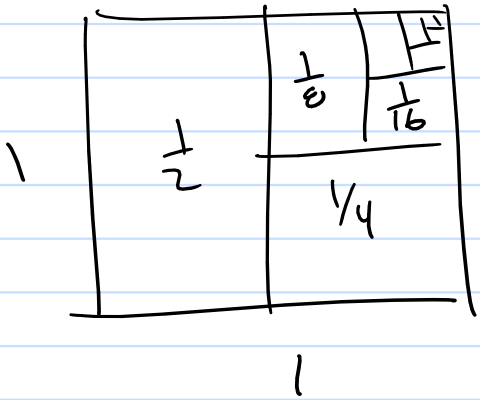
$$= \underbrace{(1+2+3+\cdots+k)}_{=\frac{k(k+1)}{2}} + (k+1)$$ by I.H.

$$= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$



$t_1 + t_2 + t_3 + t_4 + \cdots$



$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 1$

$1 + 1 + 1 + 1 + 1 + \cdots \quad \infty$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \infty$

$P(n)$: "$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$"

Prove?

$P(n=0)$: $\boxed{\text{"}1 \geq 1 + \frac{0}{2}\text{"}}$ Basis Step: is it true?

$P(n=1)$: "$1 + \frac{1}{2} \geq 1 + \frac{1}{2}$"

$P(n=2)$: "$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}} \geq 1 + \frac{2}{2}$"

$P(n=3)$: "$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq 1 + \frac{3}{2}$"

$\vdots$

$P(n=k)$: "$\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}} \geq 1 + \frac{k}{2}$"

$P(n=k+1)$: "$\boxed{\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}} \geq 1 + \frac{k+1}{2}$"

$\frac{1}{2^k+2^k} = \frac{1}{2^{k+1}}$

Finish?!