Math 415

Q's **12.** Prove that $9 \mid (4^{3n} + 8)$ for every integer $n \geq 0$.

For Induction: goal is $\forall n\, P(e_n)$

set e_i are well ordered $e_1, e_2, e_3, \ldots$

Can prove this by ..

① Basis Step: Show $P(e_1)$ is true

② Inductive Step:

Assume $P(e_1) \land \ldots \land P(e_k)$ are true (Inductive Hypothesis)

Show $P(e_{k+1})$ is true

So **12.** Prove that $9 \mid (4^{3n} + 8)$ for every integer $n \geq 0$.

Scratch..

(A)

1st element: $(n=0)$ Predicate: $9 \mid 4^{3(0)} + 8$ True or false?

2nd element: $(n=1)$ Predicate $9 \mid 4^{3(1)} + 8$ True or false?

$\vdots$

$(n=k)$ Predicate $9 \mid \boxed{4^{3k} + 8}$

$(n=k+1)$ Predicate $9 \mid 4^{3(k+1)} + 8$

is $9 \mid 4^{3k+3} + 8$

$\vdots$

(B.1) $9 \mid n$ is $9 \cdot k = n$ for integer n

(B.2) $9 \mid n$ also says n has a factor of 9.

12. Prove that $9 \mid (4^{3n}+8)$ for every integer $n \geq 0$.

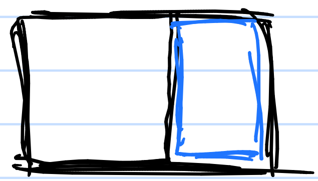
Proof: by induction we will prove the basis step of $n=0$ which is $9 \mid 4^{3\cdot 0}+8$, which is $9 \mid 9$ and this is true.

In the inductive step we will assume that $9 \mid 4^{3k}+8$ for $x = 0, 1, \ldots, k$. We will show $4^{3k+3}+8$ has a factor of 9. So,

$$4^{3k+3}+8 = 4^{3k}\cdot \underline{4^3}+8$$
$$= 64 \cdot 4^{3k} + 8 = (63+1)4^{3k}+8$$
$$= \boxed{63\cdot 4^{3k}} + \boxed{4^{3k}+8} \qquad 9 \mid 4^{3k}+8$$
$$= 9\cdot 7\cdot 4^{3k} + 9\cdot h \quad \text{for } h \in \mathbb{Z} \text{ by I.H.}$$
$$= 9(7\cdot 4^{3k}+h)$$

therefore $9 \mid 4^{3k+3}+8$

Q



30. Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

1st $f_0 = 0$, $f_1 = 1$ $\quad \underline{f_n = f_{n-1} + f_{n-2}}$; $n = 2, 3, 4, 5, \ldots$

Seq: $0, 1, 1, 2, 3, 5, 8, \ldots$

2nd golden ratio $\frac{1+\sqrt{5}}{2}$

3rd $\frac{2}{(1+\sqrt{5})}\frac{(1-\sqrt{5})}{(1-\sqrt{5})} = -\frac{1-\sqrt{5}}{2}$

30. Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$\vdots$
$F_{k-1} =$
$F_k =$
$F_{k+1} = F_k + F_{k-1}$

1st case: $(n=0)$

Basis Step

$$F_0 \stackrel{?}{=} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0}{\sqrt{5}}$$

$$0 \stackrel{?}{=} 0 \quad \text{true!}$$

Inductive Step

Assume formula works for $F_0, F_1, \ldots F_k$

$$F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

Show: $$F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

Given the definition of fib. Numbers ..

$$F_{k+1} = F_k + F_{k-1}$$

By I.H. $$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\boxed{\frac{1+\sqrt{5}}{2}}\right)^k - \left(\boxed{\frac{1-\sqrt{5}}{2}}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} = ??$$

Name: Key

Math 415 ... Exam 2

1) Prove: If a is an odd number then $a^2 + 3$ has a factor of 4.

(direct) Assume $a = 2k+1$, $k \in \mathbb{Z}$

therefore $a^2 + 3 = (2k+1)^2 + 3 = 4k^2 + 4k + 1 + 3$

$= 4(k^2 + k + 1)$ ∎

2) Prove: If 3 divides a, then 3 also divides $a^2 + 2a - 3$.

(direct) $3 | a$ means 3 is a factor of a and $3k = a$, $k \in \mathbb{Z}$.

So $a^2 + 2a - 3 = 9k^2 + 6k - 3 = 3(3k^2 + 2k - 1)$

$a^2 + 2a - 3$ has a factor of 3 so $3 | a^2 + 2a - 3$ ∎

3) Prove: For x and y integers, if $x^2(y + 3)$ is even, then x is even or y is odd.

Contrapositive: Show x is odd and y is even then $x^2(y+3)$ is odd.

Assume $x = 2k_1 + 1$, $y = 2k_2$ $k_i \in \mathbb{Z}$

So $x^2(y+3) = (2k_1+1)^2(2k_2+3)$

$= \ldots$ (finish)

$=$ we should see $2(\text{int}) + 1$

4) Prove the lemma: If 3 divides a^2, then 3 divides a.

Contrapositive $3 \nmid a \to 3 \nmid a^2$

$3 \mid a^2 \to 3 \mid a$

$3 \nmid a$ means ① $a = 3k+1$

② $a = 3k+2$

$\boxed{\text{Case 1}}$ $a = 3k+1$ so $a^2 = 9k^2 + 6k + 1$

$= 3(3k^2 + 2k) + 1$

$\boxed{\text{Case 2}}$ $a = 3k+2$ so $a^2 = 9k^2 + 12k + \underset{3+1}{4}$

$= 3(3k^2 + 4k + 1) + 1$

so $3 \nmid a^2$

∎

5) Prove: $\sqrt{3}$ is irrational.

$\boxed{\text{Contradiction}}$ assume $\sqrt{3} = \frac{a}{b}$ $a, b \in \mathbb{Z}$, $b \neq 0$, no common factors

so $3 = \frac{a^2}{b^2}$ gives $3b^2 = a^2$ means $3 \mid a^2$

by #4 we have $3 \mid a$. Because $3 \mid a$ we have

$a = 3 \cdot k$, $k \in \mathbb{Z}$. Thus $3b^2 = (3k)^2$

or $3b^2 = 9k^2$ gives $b^2 = 3k^2$

means $3 \mid b^2$ by #4 $3 \mid b$ and now

we know a, b have no common factor and

a common factor of 3. Contradiction

∎

6) Prove: $\log_3 4$ is irrational.

By contradiction: Assume $\log_3 4 = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, $b \neq 0$, no common factors.

$\log_b x = y$ means $x = b^y$

gives $4 = 3^{a/b}$ therefore $4^b = 3^a$. We know 4^b is even and 3^a is odd so we found an even = odd

Contradiction

7) Prove: $\underline{10 \text{ divides}}$ n if and only if both 2 and 5 divide n.

n has a factor of 10 $\underline{\text{iff}}$ n has a factor of 2 and a factor of 5

Pf assume n has a factor of 10

$\equiv$ " $n = 10 \cdot k$, $k \in \mathbb{Z}$ "

$\equiv$ " $n = 5 \cdot 2 \cdot k$, $k \in \mathbb{Z}$ "

$\equiv$ " n has a factor of 5 and a factor of 2"

∎

8) Prove: For integers a and b, If $(a^2 - 2a)b^2$ is odd, then a and b are odd.

(contrapositive) prove if a or b are even, then $(a^2-2a)b^2$ is even

$\boxed{\text{Case 1}}$ a is even. So $a = 2k$, $k \in \mathbb{Z}$ and $(a^2-2a)b^2 = (4k^2-4k)b^2$

$= 2(2k^2-2k)b^2$

is even

$\boxed{\text{Case 2}}$ b is even. So $b = 2k$, $k \in \mathbb{Z}$

and $(a^2-2a)b^2 = (a^2-2a)(4k^2) = 2(a^2-2a)(2k^2)$

is even

9) Prove: If two integers have the same parity, then their sum is even.

$\boxed{\text{Case 1}}$ x, y are both even integers. So $x = 2k_1$, $y = 2k_2$ $k_i \in \mathbb{Z}$

gives $x + y = 2k_1 + 2k_2 = 2(k_1 + k_2)$ is even.

$\boxed{\text{Case 2}}$ x, y are both odd ints. So $x = 2k_1+1$, $y = 2k_2+1$, $k_i \in \mathbb{Z}$.

gives $x + y = 2(k_1 + k_2 + 1)$ is even.

10) Prove: There exits a natural number n such that 5 divides $2^n - 1$.

$2^0 - 1 = 0$

$2^1 - 1 = 1$

$2^2 - 1 = 3$

$2^3 - 1 = 7$

$\boxed{2^4 - 1 = 15}$

witness

11) Use a non-constructive proof to prove: There is an irrational number raised to an irrational power that is rational.

review $\sqrt{2}^{\sqrt{2}}$ from lecture!

12) Prove: There is a unique real solution to $ax + b = c$ for $a \neq 0$.

existence: $X = \frac{c-b}{a}$ is a soln. for $a \neq 0$

proof: $a\left(\frac{c-b}{a}\right) + b = c - b + b = c$ ∎

Unique assume a second solution, call it y.

So $ay + b = c$ and $ax + b = c$ are solns.

gives $ay + b = ax + b$

$ay = ax$

$y = x$ so x is y. Have unique

∎