Math 415

Q's

29. Suppose $A \neq \emptyset$. Prove that $A \times B \subseteq A \times C$ if and only if $B \subseteq C$.

Pf Case 1: $(A \times B \subseteq \_ \times C) \Rightarrow (B \subseteq C)$

Case 2: $(B \subseteq C) \rightarrow (A \times B \subseteq A \times C)$

Facts

$S_1 \times S_2 = \{ (x,y) \mid x \in S_1 \wedge y \in S_2 \}$

$T_1 \subseteq T_2$ means $\forall e \, (e \in T_1 \rightarrow e \in T_2)$

Case 1: assume $A \times B \subseteq A \times C$ we need to show that $B \subseteq C$.

given $A \times B \subseteq A \times C$ we have all $(a,b) \in A \times B$ are also $(a,b) \in A \times C$

? get to $\forall b \; b \in B \rightarrow b \in C$?

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assume $A \times B \subseteq A \times C$ is true.

$\forall a \in A$ and $b \in B$ if $(a,b) \in A \times \boxed{B}$ then $(a,b) \in A \times \boxed{C}$

no way to remove the cross product.

Direct doesn't look to work.

$\boxed{\text{Case 1}}$ $(A \times B) \subseteq (A \times C) \rightarrow (B \subseteq C)$

Contrapositive

$$\neg(B \subseteq C) \rightarrow \neg((A \times B) \subseteq (A \times C))$$

$$\neg(\forall b\ (b \in B \rightarrow b \in C))$$

$$\exists b\ \neg(b \in B \rightarrow b \in C)$$

$$\exists b\ (b \in B \land b \notin C) \rightarrow \exists (a,b)\ ((a,b) \in A \times B \land (a,b) \notin A \times C)$$

assume and call the b not in C to be $\boxed{b^*}$

now (a,b^*) is in $A \times B$. b/c $a \in A \land b^* \in B$

but (a,b^*) is not in $A \times C$ b/c $a \in A \land b^* \notin C$.

More Stuff to Prove!

Cross Product

$$S_1 \times S_2 \times \cdots \times S_n = \{(s_1, s_2, \ldots, s_n) \mid \forall i\ s_i \in S_i\}$$

= set of all ordered n-tuples

$$|S_1 \times S_2 \times \cdots \times S_n| = |S_1|\,|S_2| \cdots |S_n|$$ = total number of all possible n-tuples

n-ary relationship is a subset of $S_1 \times S_2 \times \cdots \times S_n$

How many n-ary relationships can exist?

$$|\text{subsets}| = |P(S_1 \times S_2 \times \cdots \times S_n)| = 2^{|S_1||S_2|\cdots|S_n|}$$

ex $A = \{1, 2\}$ $\quad |A \times B \times C| = 2 \cdot 3 \cdot 2 = 12$

$B = \{a, b, c\}$

$C = \{\square, \Delta\}$ $\quad A \times B \times C = \{ (1,a,\square), (1,a,\Delta), (1,b,\square), (1,b,\Delta),$

$\cdots, (2, c, \Delta) \}$

3-ary Relationship

ex $R_1 = \{ \}$

ex $R_2 = \{ (1,a,\square), (2,b,\Delta) \}$

$$|\text{Relationships}| = 2^{12} = 2^2 \cdot 2^{10} = 4 \cdot 1024$$

Binary Relations: $R \subseteq S_1 \times S_2$

$\boxed{\text{Relation on a Set} \qquad R \subseteq S \times S}$

explain why (a,b) is here.

Representations: ① Set builder notation $R = \{ (a,b) \mid \bigcirc \}$

ex $R = \{ (x,y) \mid x, y \in \mathbb{Z} \wedge (x+1 = y) \}$

② List representation $(2,3) \in R$ b/c $2+1=3$

Notation: $(a,b) \in R$ or $(a,b) \notin R$

(use)

aRb $\qquad\qquad$ $a \not R b$

"a is related to b" $\qquad$ "a is not related to b"

③ Directed Graphs (digraph)

R on set A

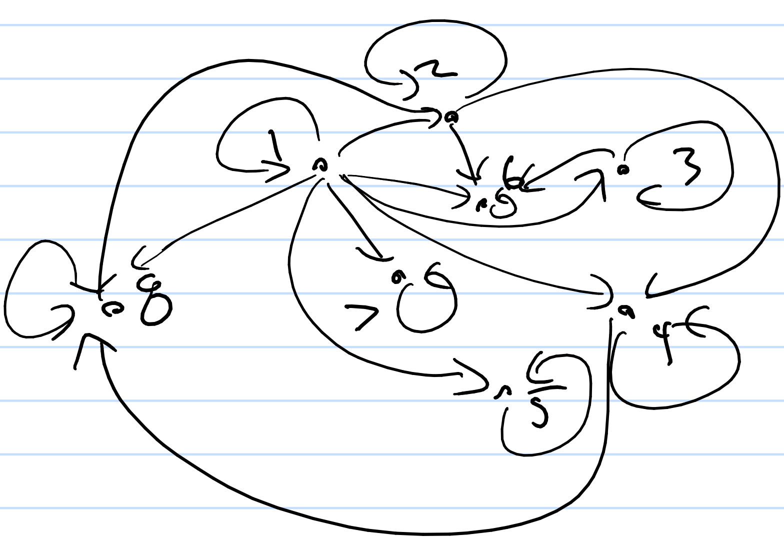
(a) all $a \in A$ are points

(b) for $(s,t) \in R$

$$s \bullet \longrightarrow \bullet\, t$$

(ex) $A = \{1,2,3,4,5,6,7,8\}$

$R = \{(x,y) \mid x \text{ is a factor of } y\}$ on A.



$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,4), (2,6), (2,8), (3,6), (4,8)\}$

④ Zero one matrix $\qquad R = [r_{ij}]$

$$r_{ij} = \begin{cases} 1 & \text{if } e_i R e_j \\ 0 & \text{if } e_i \not R e_j \end{cases}$$

(ex)

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Why Relations on set A?

(ex) $\sqrt{x^2} = |x|$

$1+2 = 3$

"Same"?

(ex) $3 > 2$

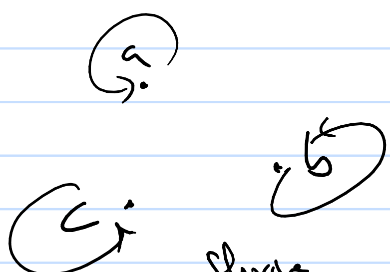
"order"?

Properties of Relations on Set A

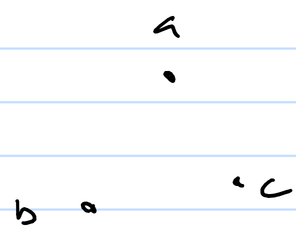
(1) a relation is reflexive if $\forall e\,(eRe)$

(2) a relation is irreflexive if $\forall e\,(e \not R e)$

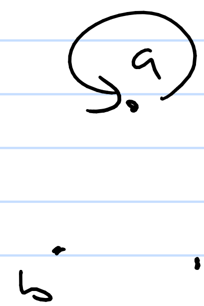
(ex) $A = \{a, b, c\}$



reflexive



irreflexive



not reflexive
not irreflexive

$$\begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$$

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(3) Symmetric $\forall a \forall b\,(aRb \rightarrow bRa)$

digraph

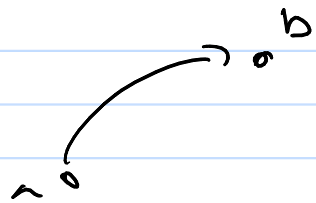
Matrix

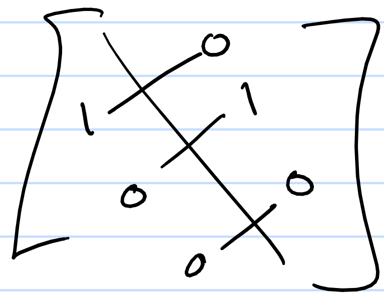
$$\begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

④ antisymmetric $\forall a \forall b \, (aRb \wedge bRc \rightarrow a = b)$

$\equiv \forall a \forall b \, (a \neq b \rightarrow (a \not R b \vee b \not R a))$

$\equiv \forall a \forall b \, (a \neq b \rightarrow \neg (aRb \wedge bRa))$



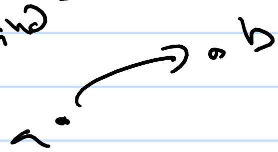


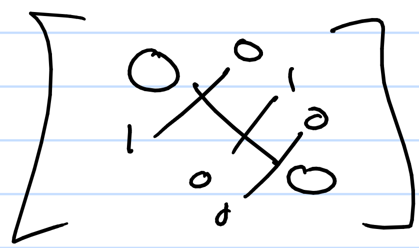
⑤ asymmetric $\forall a \forall b \, (aRb \rightarrow b \not R a)$

asymmetric $\equiv$ irreflexive $\wedge$ antisymmetric

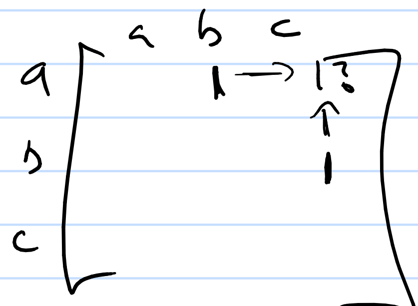
no loops

and





⑥ Transitive $\underline{\underline{\forall a \forall b \forall c}} \, (aRb \wedge bRc \rightarrow aRc)$



R on set A

$A = \{a, b, c\}$

$R = \{\ \}$

b

a c

$$M_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

reflexive? : No

irreflexive? : Yes

symmetric? : Yes $\forall a \forall b (\underbrace{aRb}_{F} \to bRa)$

antisym? : Yes

asym? : Yes

trans? : Yes

"Same"

Equivalence Relation

R is an Equiv. Relation if it is..

1. reflexive
2. symmetric
3. transitive