Math 415

Due Wed 12.1(3, 4, e)

12.2(a)

Today: ① Prove $\mathbb{Q}$ are countable

② Prove $\mathbb{R}$ are not countable

$\boxed{\text{Exam is Next Wed}}$

11.2 #2

2. Consider the relation $R = \{(a,b),(a,c),(c,c),(b,b),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

① ref? $\forall e\,(eRe)$ $\qquad A = \{a,b,c\}$

No

(a,a)? $\qquad (b,b)$? $\qquad (c,c)$?

→ No $\qquad$ yes $\qquad$ yes

counterexample.

② irref? $\forall e\,(e \not R e)$

No

$(a,a) \notin R \qquad (b,b) \notin R \qquad (c,c) \notin$?

→ counterexample $(b,b) \in R$

③ symmetric? $\forall e_1, e_2\,(e_1 R e_2 \to e_2 R e_1)$

No

if you have $(e_1, e_2) \in R$

you must have $(e_2, e_1) \in R$

b/c $(a,b) \in R$ and $(b,a) \notin R$

④ antisym? $\forall e_1 e_2\,(e_1 R e_2 \wedge e_2 R e_1 \to e_1 = e_2)$

No

$\equiv \forall e_1, e_2\,(e_1 \neq e_2 \to \neg(e_1 R e_2 \wedge e_2 R e_1))$

b/c $(c,b) \in R \wedge (b,c) \in R \wedge b \neq c$

⑤ asymmetric $\forall e_1, e_2 \, (e_1 R e_2 \rightarrow e_2 \not R e_1)$

antisymmetric $\boxed{\text{AND}}$ irreflexive.

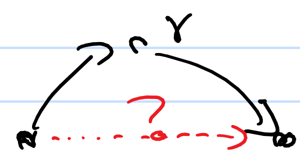
$\boxed{\text{No}}$ <u>counterexample:</u> $(b,b) \in R$

or $(b,c) \in R \land (c,b) \in R \land b \neq c$

⑥ <u>transitive</u> $\forall e_1, e_2, e_3 \, (e_1 R e_2 \land e_2 R e_3 \rightarrow e_1 R e_3)$

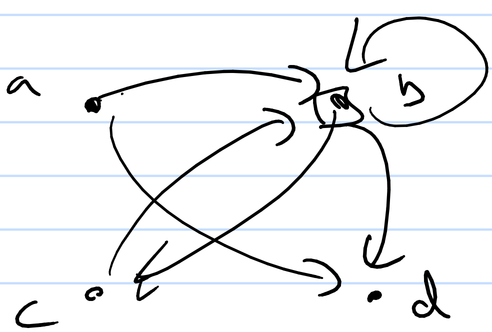
<u>No counterexample</u> found $\rightarrow$ $\boxed{\text{is transitive}}$

<u>Warshall's Algorithm</u>
- index through all $e \in A$
- check if vertex is a middle



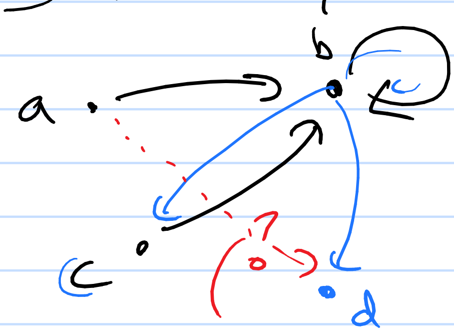
- for all edges to v followed by all edges from v

check if trans edge exists



$\boxed{a}$ Middle? no

$\boxed{b}$ Middle? yes



$(a,d) \notin R$ is a counterexample

11.3 #8

8. Define a relation R on $\boxed{\mathbb{Z}}$ as xRy if and only if $\boxed{x^2 + y^2 \text{ is even}}$. Prove R is an equivalence relation. Describe its equivalence classes.

Is R (1) ref?. $\forall e\,(eRe)$ is e^2+e^2 needs to be even

(2) sym?. $\forall e_1 e_2\,(e_1Re_2 \to e_2Re_1)$

(3) trans?. $\forall e_1 e_2 e_3\,(e_1Re_2 \wedge e_2Re_3 \Rightarrow e_1Re_3)$

$R = \{(x,y) \mid$ all $x,y \in \mathbb{Z}$ and x^2+y^2 is even$\}$

$(3,5) \in R$

b/c $3^2+5^2 = 9+25 = 34$ is even

$(3,2) \notin R$

$3^2+2^2 = 9+4 = 13$ is not even.

$(3,3) \in R$

$3^2+3^2 = 9+9 = 2\cdot 9$

$(10,10) \in R$

$10^2+10^2 = 2\cdot 10^2$ even!

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trans?: if $(e_1^2+e_2^2)$ is even

and $(e_2^2+e_3^2)$ is even

show $e_1^2+e_3^2$ is even.

Idea? (1) Discuss parity?

(2) factors of 2?. $2 \mid (e_1^2+e_2^2)$

$2 \mid (e_2^2+e_3^2)$

Facts: $a \mid b \wedge a \mid c \to a \mid mb+nc$

ex $2 \mid 6 \wedge 2 \mid 8 \to 2 \mid (6-8)$

equiv classes: $[e]_R = \{x \mid eRx\}$

$(2,4) \in R$ b/c $2^2+4^2 = 4+16 = 20$ is even.

why? $[2] = \{\dots,-4,-2,0,2,4,\dots\}$ all evens

$[1] = \{\dots,-3,-1,1,3,\dots\}$

b/c $(2,x) \in R$ says 2^2+x^2 is even

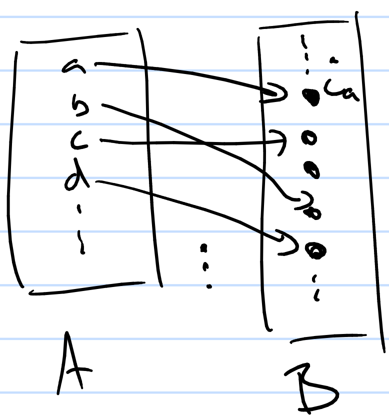
even number

so call $2^2+x^2 = E$

$x^2 = E - 2^2$

even even

Functions Injection (one-to-one)



$\forall a,b$ if $[f(a)=f(b) \to a=b]$

or

$\forall a,b$ if $[a \neq b \to f(a) \neq f(b)]$

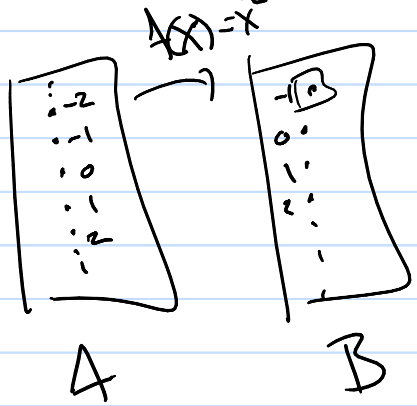
$a,b \in A$

show injective? (prove the above implication)

show not one-to-one? $\exists a,b\,(f(a)=f(b) \wedge a \neq b)$

Surjective (onto) $\forall b \in B \ \exists a \in A \ f(a) = b$

(also says Range = Codomain)



bijective: both injective and surjective.

Cardinalities

bijection

$|A| = |B|$