Math 415

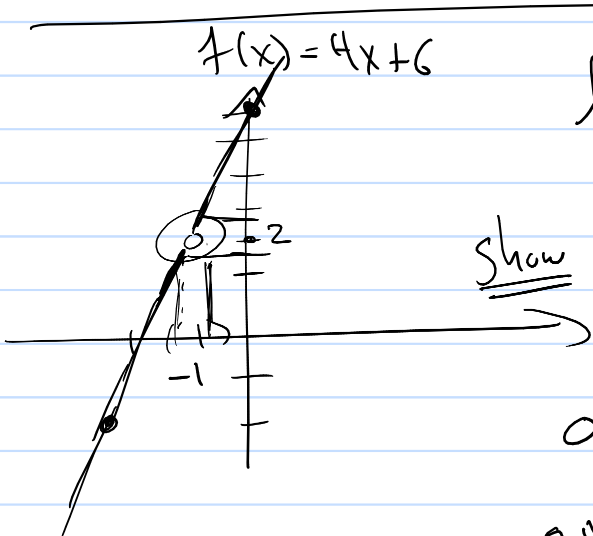
Q's

2. Prove that $\lim_{x\to -1}(4x+6)=2$.
4. Prove that $\lim_{x\to 8}(2x-7)=9$.
6. Prove that $\lim_{x\to 1}(4x^2+1)=5$.

Calc 1

$\lim_{x\to c} f(x) = f(c)$ if $f(x)$ is cont.

$$\lim_{x\to -1} 4x+6 = 4(-1)+6 = 2$$



$$\lim_{x\to -1} 4x+6 = 2$$

Show for all real numbers $\varepsilon>0$ there is a real number $\delta>0$ such that

$$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

gives $\lim_{x\to c} f(x) = L$

So ε-δ proofs:

Assume $0<|x-c|<\delta$ is true (we need to $\boxed{\text{know}}$ δ)

and show $|f(x)-L|<\varepsilon$ is always true.

For this problem $\lim_{x\to -1} 4x+6 = 2$

$\boxed{\text{is}}$ $\boxed{\text{prove}}$ $0<|x+1|<\delta \rightarrow |(4x+6)-2|<\varepsilon$

Scratch $\boxed{0 < |x+1| < \delta}$ $\rightarrow$ $\boxed{|(4x+6) - 2| < \varepsilon}$

I.H. but

Backwards reasoning?

we need δ to be selected based on ε.

(function of ε)

by observation

$\delta = \frac{1}{4}\varepsilon$

$|4x+6-2| < \varepsilon$

$|4x+4| < \varepsilon$

$|4(x+1)| < \varepsilon$

$|4||x+1| < \varepsilon$

$4|x+1| < \varepsilon$

$\boxed{|x+1| < \varepsilon/4}$

$\boxed{\text{pf}}$ assume $0 < |x+1| < \delta$. Let $\delta = \frac{1}{4}\varepsilon$ so

$|x+1| < \frac{1}{4}\varepsilon$

which is $4|x+1| < \varepsilon$

" " $|4x+4| < \varepsilon$

" " $|4x+6-2| < \varepsilon$

" $|(4x+6)-2| < \varepsilon$

therefore if $0 < |x-(-1)| < \delta$ with $\delta = \frac{1}{4}\varepsilon$

gives $|(4x+6)-2| < \varepsilon$

by def $\lim_{x \to -1} 4x+6 = 2$

$\blacksquare$

2. Prove that $\lim_{x\to -1}(4x+6)=2$.

4. Prove that $\lim_{x\to 8}(2x-7)=9$.

6. Prove that $\lim_{x\to 1}(4x^2+1)=5$.

$\delta = ?$

ε-δ $\quad \boxed{0<|x-1|<\delta} \longrightarrow |(4x^2+1)-5|<\varepsilon$

Scratch

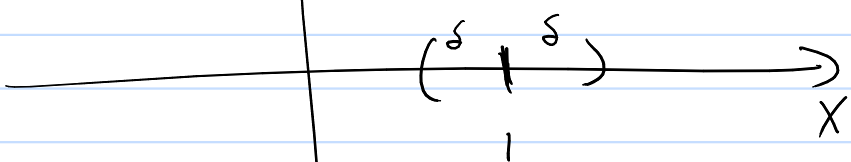
$$\boxed{|(4x^2+1)-5|<\varepsilon}$$

$$|4x^2-4|<\varepsilon$$

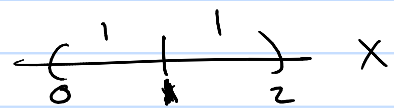
$$4|x^2-1|<\varepsilon \longrightarrow \boxed{4|x+1||x-1|<\varepsilon}$$

$$|x-1|<\delta$$

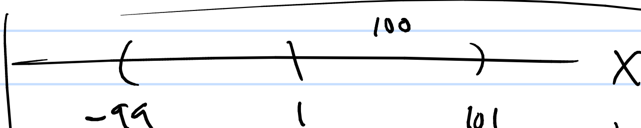
$$|x+1|$$



let $\boxed{\delta = \begin{cases} 1 & \text{as a max} \\ \text{smaller than } 1 & \end{cases}}$

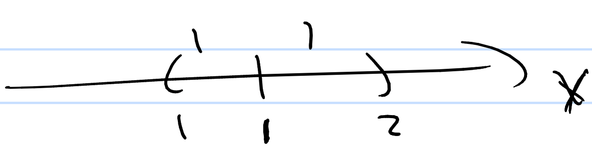


$|x+1|<3$



$|x+1|<101$

by doing this $|x+1|<3$



Back to goal $\quad 4|x+1||x-1|<\varepsilon$

Start on left $\quad 4\boxed{|x+1|}|x-1| < 4\cdot \boxed{3}|x-1|$

Found $\quad |(4x^2+1)-5| < \boxed{12|x-1| < \varepsilon}$

$<\varepsilon$

let $\delta = \frac{1}{12}\varepsilon$?

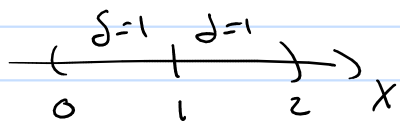
$\lim_{x \to 1} 4x^2 + 1 = 5$

Pf assume $0 < |x-1| < \delta$, let $\delta = \min\left(\frac{1}{12}\varepsilon, 1\right)$

Now, show $|(4x^2+1) - 5| < \delta$

$|(4x^2+1) - 5| = |4x^2 - 4| = 4|x+1|\,|x-1|$

Because $|x-1| < \delta$ and $\delta = 1$ at most, then

x is bounded 

between 0 and 2 for our δ. So $|x+1| < 3$

this gives $4|x+1|\,|x-1| < 4 \cdot 3\,|x-1|$

for smaller ε's let $\delta = \frac{1}{12}\varepsilon$ and by I.H. $|x-1| < \delta$

gives $12|x-1| < 12\frac{\varepsilon}{12} = \varepsilon$

and we have shown $|(4x^2+1) - 5| < \varepsilon$

∎

today review keys exam 3/4

Wed exam 5 take home (review for final)

→ old proofs from exams 2 to 4.] 6

→ ① Prove $\mathbb{Q}$ is countable

② Prove $\mathbb{R}$ is uncountable

③ Prove a set is countable (find bijection)

④ ε-δ proofs

⑤ ε-δ proofs

Monday 3pm – 4^{50} pm Final Exam.

10 "important" proofs.

NAME: Key

MATH 415 ... EXAM 3

$n = 1, 2, 3, 4, \ldots$ basis

1) Prove that for every positive integer n,

$$P(n): \text{"}\ 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}\ \text{"}$$

Proof by Induction

$\boxed{\text{Basis Case}}$ for $n=1$, $\boxed{1\cdot 2\cdot 3 = \frac{1\cdot 2\cdot 3\cdot 4}{4}}$ is true.

$\boxed{\text{Inductive Case}}$ Assume I.H. where $\boxed{1\cdot 2\cdot 3 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}}$

and we need to show $[1\cdot 2\cdot 3 + \cdots + k(k+1)(k+2)] + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$

By using I.H. $(1\cdot 2\cdot 3 + \cdots + k(k+1)(k+2)) + (k+1)(k+2)(k+3)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

by factoring $= (k+1)(k+2)(k+3)\left[\frac{k}{4} + 1\right] = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$

∎

2) Prove that $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \ldots + n\cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Proof by Induction

$\boxed{\text{Basis Case}}$ for $n=1$, $1\cdot 1! = (2)! - 1$ is $1 = 1$ which is true.

$\boxed{\text{Inductive Case}}$ Assume I.H. where $1\cdot 1! + 2\cdot 2! + \cdots + k\cdot k! = (k+1)! - 1$

and we need to show $1\cdot 1! + \cdots + k\cdot k! + (k+1)(k+1)! = (k+2)! - 1$

By using I.H. $((1\cdot 1!) + \cdots + k\cdot k!) + (k+1)(k+1)!$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

by factoring $= (k+1)!\,[1 + (k+1)] - 1 = (k+2)(k+1)! - 1$

by def. of factorial $= (k+2)! - 1$

∎

3) Prove that if $n > 4$, then $2^n > n^2$. $n = 1, 2, 3, 4, \ldots$ Prove by Induction

Basis Case for $n=5$ $2^5 > 5^2$ is $32 > 25$ is true.

Inductive Case Assume I.H. where $\boxed{2^k > k^2}$ and $\underline{k > 4}$

and we need to show $2^{k+1} > (k+1)^2$

For $(k+1)^2 = k^2+2k+1 < k^2+2k+k$ by I.H. of $k>4$

and $k^2+2k+k = k^2+3k < k^2+k \cdot k$ by I.H.

and $k^2+k \cdot k = k^2+k^2 < 2^k+2^k$ by I.H.

Therefore $(k+1)^2 < 2^{k+1}$

∎

4) Use mathematical induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Prove by Induction

Basis Case $n=1$, $(1)^3+2(1) = 3$ and $3|3$ so true.

Inductive Case Assume I.H. where $3 | k^3+2k$ and we need to show $3 | (k+1)^3+2(k+1)$.

Given $(k+1)^3 + 2(k+1) = k^3+3k^2+3k+1+2k+2$

$= k^3+3k^2+5k+3 = (k^3+2k) + (3k^2+3k+3)$

$= (k^3+2k) + 3(k^2+k+1)$

because $3 | k^3+2k$ (by inductive hyp.) and $3 | 3(k^2+k+1)$

then $3 | (k+1)^3+2(k+1)$

∎

Note: Fib. Numbers $F_0=0, F_1=1, F_2=1, F_3=2, \ldots$

5) Prove that $F_{n-1} \cdot F_{n+1} - F_n^2 = (-1)^n$ when n is a positive integer. Prove by Induction.

$\boxed{\text{Basis Case}}$ for $n=1$, $F_0 \cdot F_2 - F_1^2 = (-1)^1$ is $0 \cdot 1 - 1^2 = -1$ is true

$\boxed{\text{Inductive Case}}$ Assume I.H. $F_{k-1} F_{k+1} - F_k^2 = (-1)^k$

and we need to show $F_k F_{k+2} - F_{k+1}^2 = (-1)^{k+1}$

Given $F_k \boxed{F_{k+2}} - F_{k+1}^2 = F_k \boxed{[F_k + F_{k+1}]} - F_{k+1}^2$

$= F_k^2 + F_k F_{k+1} - F_{k+1}^2 = F_k^2 - F_{k+1}[F_{k+1} - F_k]$

$= F_k^2 - F_{k+1} F_{k-1} = -(F_{k-1} F_{k+1} - F_k^2)$

by I.H. $= -(-1)^k = (-1)^{k+1}$

$F_0 = 0$
$F_1 = 1$
$\vdots$

6) Prove that $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \cdot F_{n+1}$ for $F_1, F_2, \ldots$

Prove by Induction.

$\boxed{\text{Basis Case}}$ $n=1$, $F_1^2 = F_1 \cdot F_2$ is $1^2 = 1 \cdot 1$ is true

$\boxed{\text{Inductive Case}}$ Assume I.H. $F_1^2 + F_2^2 + \cdots + F_k^2 = F_k F_{k+1}$

and we need to show $F_1^2 + \cdots + F_k^2 + F_{k+1}^2 = F_{k+1} F_{k+2}$.

Given $(F_1^2 + F_2^2 + \cdots + F_k^2) + F_{k+1}^2 = F_k F_{k+1} + F_{k+1}^2$ by I.H.

and with factoring $F_k F_{k+1} + F_{k+1}^2 = F_{k+1}[F_k + F_{k+1}] = F_{k+1} F_{k+2}$

$F_0, F_1, F_2, F_3, F_4, \ldots, F_{k-2}, F_{k-1}, F_k, F_{k+1}, \ldots$

$F_n = F_{n-1} + F_{n-2}$

7) For the sets $A_1, A_2, \ldots A_n$ in the same universe of discourse prove that $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$, when $n \geq 2$.

Prove by Induction.

$\boxed{\text{Basis Case}}$ for $n=2$ $\boxed{\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}}$ is De Morgan's Law. So, true

$\boxed{\text{Inductive Case}}$ Assume I.H. where $\overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$.

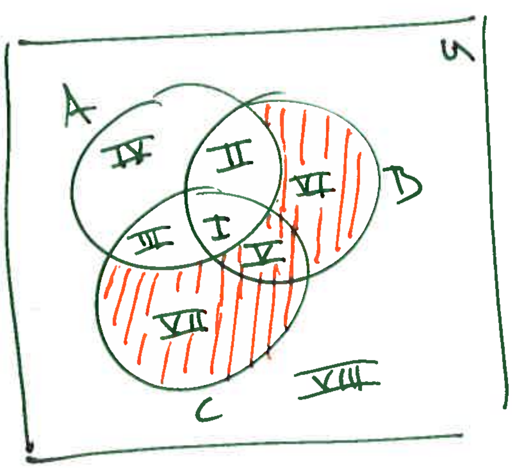
and we need to show $\overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup (A_{k+1})} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}$.

By grouping we have $\overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} = (\overline{A_1 \cup A_2 \cup \cdots \cup A_k}) \cap (\overline{A_{k+1}})$

and by I.H. $= \underbrace{\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}}_{\text{I.H.}} \cap \overline{A_{k+1}}$

8) Use a membership table to verify $(B - A) \cup (C - A) = (B \cup C) - A$ and draw the Venn Diagram for its regions.

region	A	B	C	B-A	C-A	(B-A)∪(C-A)	(B∪C)	(B∪C)-A
I	1	1	1	0	0	0	1	0
II	1	1	0	0	0	0	1	0
III	1	0	1	0	0	0	1	0
IV	1	0	0	0	0	0	0	0
V	0	1	1	1	1	1	1	1
VI	0	1	0	1	0	1	1	1
VII	0	0	1	0	1	1	1	1
VIII	0	0	0	0	0	0	0	0



Both $(B-A) \cup (C-A)$ and $(B \cup C)-A$ are regions 5, 6, 7 of the Venn diagram.

9) Prove that $(A - B) - C \subseteq (A - C)$. Note: the rule of inference called simplification is the tautology $(a \wedge b) \rightarrow a$.

Direct Proof:

Assume an element $e \in ((A-B)-C)$. Therefore $e \in (A-B) \wedge e \notin C$ which is logically $e \in A \wedge e \notin B \wedge e \notin C$. By simplification we have $e \in A \wedge e \notin C$. which is $e \in (A-C)$. Which shows for $e \in ((A-B)-C)$, then $e \in (A-C)$

or $((A-B)-C) \subseteq (A-C)$

∎

10) Prove that $A \times (B - C) = (A \times B) - (A \times C)$. Start with right side...

$(A \times B) - (A \times C) = \{ (x,y) \mid (x,y) \in A \times B \wedge \neg((x,y) \in A \times C) \}$

$= \{(x,y) \mid (x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C)\}$

$= \{(x,y) \mid (x \in A \wedge y \in B) \wedge (x \notin A \vee y \notin C)\}$

$= \{(x,y) \mid (x \in A \wedge y \in B \wedge x \notin A) \vee (x \in A \wedge y \in B \wedge y \notin C)\}$

$= \{(x,y) \mid \quad F \quad \vee (x \in A \wedge y \in B \wedge y \notin C)\}$

$= \{(x,y) \mid x \in A \wedge (y \in B \wedge y \notin C)\}$

$= \{(x,y) \mid x \in A \wedge y \in (B-C)\}$

$= A \times (B-C)$

11) Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$. Note: this is just one possible proof.

Case 1: $A \subseteq B \rightarrow P(A) \subseteq P(B)$

Assume I.H. that $A \subseteq B$. We want to show $P(A) \subseteq P(B)$.

Now for any $S \subseteq P(A)$ it means $S \subseteq A$. By I.H. of $A \subseteq B$ we have $S \subseteq B$. By def of a power set $S \in P(B)$ and we have shown $P(A) \subseteq P(B)$.

Case 2: $P(A) \subseteq P(B) \rightarrow A \subseteq B$

Assume I.H. that $P(A) \subseteq P(B)$. We know $A \subseteq A$ and therefore $A \in P(A)$. By I.H. of $P(A) \subseteq P(B)$ anything in $P(A)$ is in $P(B)$ and therefor $A \in P(B)$.

To be an element of a power set means it is a subset, therefore $A \in P(B) \rightarrow A \subseteq B$.

$\blacksquare$

NAME: Key

MATH 415 ... EXAM 4

1) Is the relation r consisting of all ordered pairs (a, b) such that a and b are humans and have at least one common genetic parent: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive? Check all the properties and if a property doesn't hold give a counter-example. Also, **state the logical definitions of the properties** as you consider them.

Reflexive: $\forall a\,(aRa)$ says every person has at least one common genetic parent with themself. True, because we have two genetic parents.

Irreflexive: $\forall a\,(a \not R a)$ says no one has a common genetic parent with themself. False, Counter example is me. (or any human)

Symmetric: $\forall a,b\,(aRb \to bRa)$ says a person share a common genetic parent with another person then that person shares a common genetic parent with the first. True, because having at least one genetic parent will be the tie between any related two people in this relation.

antisymmetric: $\forall a,b\,(aRb \wedge bRa \to a=b)$ says for different people they will have no common genetic parent or one person share at least one genetic parent with the other but the other person does not. False, Counter example is me and my brother. We are different people and share two common genetic parents.

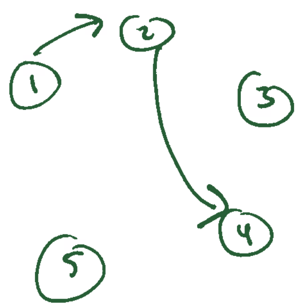
asymmetric, $\forall a,b\,(aRb \to b \not R a)$ says if one person shares at least one common genetic parent with another then the other does not share a common genetic parent with the first. False, Counter example is me and my brother.

transitive. $\forall a,b,c\,(aRb \wedge bRc \to aRc)$ says if two people share at least one common genetic parent and the second shares at least one with a third, the first and third do as well. False, Counter example is step siblings with only one common parent.

2) Given the relation $R = \{(a, b) | b = 2a\}$ on the set of positive integers from -2 to 5. Give the list of ordered pairs for R and represent is as a digraph. Also, determine if it is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive? Check all the properties and if a property doesn't hold give a counter-example.

Set = $\{1, 2, 3, 4, 5\}$

So $R = \{ (1,2), (2,4) \}$



(a) it is not reflexive b/c $(1,1) \notin R$

(b) it is irreflexive b/c for all a in the Set $(a,a) \notin R$.

(c) it is not symmetric b/c $(2,1) \notin R$

(d) it is antisymmetric b/c $(2,1) \notin R$ and $(4,2) \notin R$.

(e) it is asymmetric b/c it is irreflexive and antisym.

(f) it is not transitive b/c $(1,2) \in R \wedge (2,4) \in R$ and $(1,4) \notin R$.

3) Show that the relation R consisting of all pairs of polynomials (f, g) such that the first derivative of f and the first derivative of g are equal is an equivalence relation on the set of all polynomials with real-valued coefficients.

(a) R is reflexive b/c $f' = f'$ and hence $(f, f) \in R$ for all f.

(b) R is symmetric b/c if $f' = g'$ then $g' = f'$ and therefore if $(f, g) \in R \Rightarrow (g, f) \in R$

(c) R is transitive b/c if $f' = g'$ and $g' = h'$ then $f' = h'$ and therefore $(f, g) \in R \wedge (g, h) \in R \Rightarrow (f, h) \in R$.

4) Verify that the relation $R = \{(x, y) | 2 \text{ divides } x^2 + y^2\}$ on the set of all integers is an equivalence relation. Describe its equivalence classes.

Note: $2 | x^2+y^2$ says x^2+y^2 is even.

(a) R is reflexive b/c $x^2+x^2 = 2x^2$ is even and therefore $(x,x) \in R$.

(b) R is symmetric b/c if x^2+y^2 is even then y^2+x^2 is even.

(c) R is transitive b/c if x^2+y^2 is even and y^2+z^2 is even

then x^2, y^2, z^2 are all of same parity.

and then x^2+z^2 is even.

For any $(x,y) \in R$ x^2+y^2 is even or x^2, y^2 are same parity. But x^2, x have same parity.

and y^2, y have same parity.

Therefore we have equivalence classes based on parity.

$[a]$ = set of all odds

and

$[b]$ = set of all evens.

5) Prove Theorem 11.1 on page 215 of Book of Proof.

6) Is the relation $R = \{(x, y) | x^2 + y^2 = 1\}$ a function on the Real Numbers?

No. Consider $x=0$ then we have $0^2 + y^2 = 1$ or $y = 1$ or $y = -1$

so $(0,1) \in R$ and $(0,-1) \in R$. Counter example to

R being a function.

7) ~~Is the~~ relation $R = \{(s, n) | s$ is any string and n is the number of characters in $s\}$ a function from the set of strings to the set of integers?

For any string $S = c_1 c_2 \cdots c_n$ a seq of n characters. Only λ

the empty string will have $(\lambda, 0)$. All other strings are

$(c_1 c_2 c_3 \ldots c_n, n) \in R$. And because length is unique we have only

one n for all strings. Therefore R is a function.

8) Is the function given in problem (7) injective?

No. Counter example ('Mark', 4) and ('John', 4).

9) Is the function given in problem (7) surjective?

No. Because length is never negative.

10) Give an example of a bijection from the rational numbers to the positive integers.

Consider the table made of 0 and $\pm\frac{a}{b}$ $a, b \in \mathbb{Z}^+$, ...

$d=1$ $d=2$ $d=3$ $d=4$

0

$\pm\frac{1}{1}$ $\pm\frac{2}{1}$ $\pm\frac{3}{1}$ ...

$\pm\frac{1}{2}$ $\pm\frac{2}{2}$ $\pm\frac{3}{2}$ ...

$\pm\frac{1}{3}$ $\pm\frac{2}{3}$ $\pm\frac{3}{3}$ ...

$\vdots$ $\vdots$ $\vdots$ $\ddots$

Walking through the diagonals $d = 1, 2, 3, \ldots$ we see each diagonal is of finite length and the table includes all rationals plus some non-rational fractions like $\pm 4/2$, etc when a, b have common factors. Also each diagonal contains any $\frac{a}{b}$ s.t. $d = a+b$. So form the bijection by walking the diagonals and counting only the rationals.

11) For the sets of $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. all ordered pairs are $|A||B| = 4 \cdot 3 = 12$

a) How many relations from A to B?

a relation is a subset. So one of $\boxed{2^{12}}$ possible relations (subsets)

b) How many functions from A to B?

$\boxed{3^4}$. Because 3 choices for every $e \in A$.

c) How many injections from A to B?

$\boxed{\text{None}}$. Because $|A| > |B|$

d) How many ~~surjections from A~~ to B?

$\boxed{\dfrac{4 \cdot 3 \cdot 2 \cdot 3}{2}}$ is 4 arrows to 1, 3 to 2, 2 to 3 and last will form a two way symmetry. So Mult by 3 and divide by 2.

e) How many bijections from A to B?

$\boxed{\text{None}}$ Because no injections.

$\dfrac{4 \cdot 3 \cdot 2 \cdot 3}{2}$

100pts = 100%