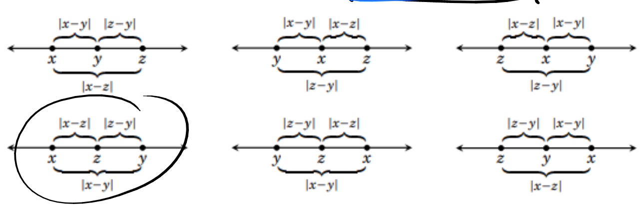
Math 415

Q's

Theorem 13.1 (Triangle inequality) If $x, y, z \in \mathbb{R}$, then $|x-y| \le |x-z| + |z-y|$.

Proof. The name *triangle inequality* comes from the fact that the theorem can be interpreted as asserting that for any "triangle" on the number line, the length of any side never exceeds the sum of the lengths of the other two sides. Indeed, the distance between any two numbers $a, b \in \mathbb{R}$ is $|a-b|$. With this in mind, observe in the diagrams below that regardless of the order of x, y, z on the number line, the inequality $\boxed{|x-y| \le |x-z| + |z-y|}$ holds.



(These diagrams show x, y, z as distinct points. If $x = y$, $x = z$ or $y = z$, then $|x-y| \le |x-z| + |z-y|$ holds automatically.) ■

The triangle inequality says the shortest route from x to y avoids z unless z lies between x and y. Several useful results flow from it. Put $\boxed{z = 0}$ to get

$$|x-y| \le |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.1}$$

Using the triangle inequality, $|x+y| = |x-(-y)| \le |x-0| + |0-(-y)| = |x| + |y|$, so

$$|x+y| \le |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.2}$$

Also by the triangle inequality $\boxed{|x-0| \le |x-(-y)| + |-y-0|}$, which yields

$$|x| - |y| \le |x+y| \quad \text{for any } x, y \in \mathbb{R}. \tag{13.3}$$

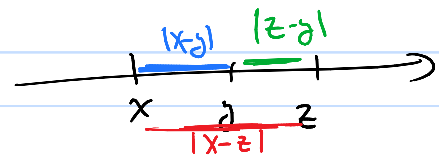
The three inequalities (13.1), (13.2) and (13.3) are very useful in proofs.

let $z=0$

let

For any $x, y, z \in \mathbb{Z}$ there are 6 cases of order.

case 1 $\qquad x \le y \le z$

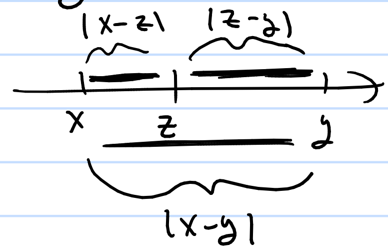


now consider $|x-y|$ vs $|x-z|$ vs $|z-y|$

$|x-y| \le |x-z|$ ~~+ |z-y|~~

so $|x-y| \le |x-z| + |z-y|$

case 4 $\qquad x \le z \le y$



$|x-z| + |z-y| = |x-y|$

so $|x-y| \le |x-z| + |z-y|$

Corr

given $\boxed{\text{thm}}$ $|x-y| \le |x-z| + |z-y|$ for all $x, y, z \in \mathbb{R}$

ex

let $x = -y$ and $z = 0$

$|-y-y| \le |-y-0| + |0-y|$

$\boxed{14.1 \ \#4}$ $|\text{set of evens}| = |\text{set of odds}|$

Means we need a bijection between the sets.

evens	f	odds
-6	$\longrightarrow$	-5
-4	$\longrightarrow$	-3
-2	$\longrightarrow$	-1
0	$\longrightarrow$	1
2	$\longrightarrow$	3
4	$\longrightarrow$	5
$\vdots$	$\vdots$	$\vdots$

$$\boxed{f(e) = e+1}$$

from evens to odds

Set of evens $= \{ e \mid e = 2k, \ k \in \mathbb{Z} \}$ &

set of odds $= \{ o \mid o = 2k+1, \ k \in \mathbb{Z} \}$

$\boxed{\mathbb{R} \text{ is uncountable}}$

$\boxed{\text{Th}^{\text{m}}}$ for $A \subseteq B$

① if A is uncountable, then B is uncountable

② if B is countable, then A is countable

So it is enough to show the reals from 0 to 1 are uncountable.

$\boxed{\text{lemma}}$ reals from 0 to 1 are uncountable.

$\boxed{\text{proof}}$ We will use a proof by contradiction. And we will assume that this set is countable.

$\boxed{\text{state Def of countability}}$

then

$$1 \to r_1 = 0.d_{11}\,d_{12}\,d_{13} \ldots$$
$$2 \to r_2 = 0.d_{21}\,d_{22}\,d_{23} \ldots$$
$$3 \to r_3 = 0.d_{31}\,d_{32}\,d_{33} \ldots$$
$$\vdots$$

is a bijection from $\mathbb{Z}^+$ to our set. Where d_{ij} is the j^{th} decimal of the i^{th} real in our ordered set.

We need uniqueness — explain the 9 problem.

(ex) $0.13999\ldots = 0.14$

and handle the problem.

Consider a special real r^*.

where $r^* = 0.d_1 d_2 d_3 d_4 \ldots$ with $d_i \in \{4, 6\}$

Obviously r^* is in our set. And we form the d_i by comparing to each r_i and select $d_i \neq d_{ii}$

i^{th} dec of r^* — i^{th} dec of r_i

Final Exam   Monday @ 3pm

3 problems per exam (of exams 1 to 5)

$\boxed{15 \text{ total}}$!

Name:

Math 415 … Exam 1

1) Which of the following are propositions?

a) Wichita is the capital of Kansas.

b) $1 + 2 = 3$

c) $x + 2 = 3$

d) Read this carefully.

e) What time is it?

2) Construct the truth table everyone should know.

3) Construct a truth table for $\neg q \to p$. Under what conditions is it false?

? 4) Construct a truth table for $c \to (r \wedge \neg a)$ Under what conditions is it false?

1 truth table type prob.

of questions.

? 5) Construct a truth table for $\neg(p \vee q) \iff (\neg p \wedge \neg q)$ Under what conditions is it false?

6) Express “For the mouse to defeat the cat it is sufficient that the mouse drinks lots of coffee” using propositional symbols and logical operators. Then contruct a truth table for your compound proposition. Under what conditions is is false?

7) Let $S(u)$ mean that “u is silly,” $F(v)$ mean that “v is fast,” and $B(a, b)$ mean that “a has beat b in a race”, where the universe of discourse for every variable consists of all children. Express $\exists x(F(x) \wedge \forall y(S(y) \rightarrow B(x, y)))$ by a simple English sentence. Under what conditions would this be false?

8) Use a truth table to check if the statements $(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

0

Verify

comp. prop $\equiv$ comp. prop.

9) Use logical equivalences to show that $(p \wedge q) \to p$ is a tautology.

10) Use logical equivalences and a truth table to verify Modus Tollens $((p \to q) \wedge \neg q) \to \neg p$.

11) Come up with valid conclusions for the set of premises: “If I eat at bedtime, then I can not sleep.” “I can not sleep if there is music playing.” “I slept last night.” “Not sleeping is sufficient for me to not pass Math 415.” Explain your answers.

12) Is the following argument valid? "You do not do every problem in the book or you learn Calculus. You learned Calculus. Therefore, you did every problem in the book." Explain.

6

Name:

Math 415 ... Exam 2

1) Prove: If a is an odd number then $a^2 + 3$ has a factor of 4.

2) Prove: If 3 divides a, then 3 also divides $a^2 + 2a - 3$.

3) Prove: For x and y integers, if $x^2(y + 3)$ is even, then x is even or y is odd.

4) Prove the lemma: If 3 divides a^2, then 3 divides a.

contrapositive followed by cases

5) Prove: $\sqrt{3}$ is irrational.

contradiction.

6) Prove: $\log_3 4$ is irrational.

7) Prove: 10 divides n if and only if both 2 and 5 divide n.

$\longleftrightarrow$

case 1 $\longrightarrow$

case 2 $\longleftarrow$

8) Prove: For integers a and b, If $(a^2 - 2a)b^2$ is odd, then a and b are odd.

step1 contraposition

step2 cases

~~9)~~ Prove: If two integers have the same parity, then their sum is even.

10) Prove: There exits a natural number n such that 5 divides $2^n - 1$.

11) Use a non-constructive proof to prove: There is an irrational number raised to an irrational power that is rational.

~~12)~~ Prove: There is a unique real solution to $ax + b = c$ for $a \neq 0$.

Name:

Math 415 ... Exam 3

26

1) Prove that for every positive integer n,

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

2) Prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

?/6 3) Prove that if $n > 4$, then $2^n > n^2$.

(induction)

?/6 4) Use mathematical induction to prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

5) Prove that $F_{n-1} \cdot F_{n+1} - F_n^2 = (-1)^n$ when n is a positive integer.

6) Prove that $F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \cdot F_{n+1}$

7

6

7) For the sets $A_1, A_2, \ldots A_n$ in the same universe of discourse prove that $\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$, when $n \geq 2$.

(Induction)

8) Use a membership table to verify $(B - A) \cup (C - A) = (B \cup C) - A$ and draw the Venn Diagram for its regions.

7

9) Prove that $(A - B) - C \subseteq (A - C)$.

6

10) Prove that $A \times (B - C) = (A \times B) - (A \times C)$.

7

6

11) Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

Name:

Math 415 ... Exam 4

7

1) Is the relation r consisting of all ordered pairs (a, b) such that a and b are humans and have at least one common genetic parent: reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive? Check all the properties and if a property doesn't hold give a counter-example. Also, **state the logical definitions of the properties** as you consider them.

6

2) Given the relation $R = \{(a, b)|b = 2a\}$ on the set of positive integers from -2 to 5. Give the list of ordered pairs for R and represent is as a digraph. Also, determine if it is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive? Check all the properties and if a property doesn't hold give a counter-example.

6

7

3) Show that the relation R consisting of all pairs of polynomials (f, g) such that the first derivative of f and the first derivative of g are equal is an equivalence relation on the set of all polynomials with real-valued coefficients.

6

4) Verify that the relation $R = \{(x, y) | 2 \text{ divides } x^2 + y^2\}$ on the set of all integers is an equivalence relation. Describe its equivalence classes.

5) Prove Theorem 11.1 on page 215 of Book of Proof.

6) Is the relation $R = \{(x, y)|x^2 + y^2 = 1\}$ a function on the Real Numbers?

7) Is the relation $R = \{(s, n)|s$ is any string and n is the number of characters in $s\}$ a function from the set of strings to the set of integers?

7
6

8) Is the function given in problem (7) injective?

9) Is the function given in problem (7) surjective?

10) Give an example of a bijection from the rational numbers to the positive integers.

11) For the sets of $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$.

a) How many relations from A to B?

b) How many functions from A to B?

c) How many injections from A to B?

d) How many surjections from A to B?

e) How many bijections from A to B?

Name:

Math 415 ... Exam 5

1) Prove that for all integers x, then x and x^2 have the same parity.

2) Prove that $\sqrt{5}$ is irrational.

3) Prove if two integers have opposite parity, then their sum is odd.

4) Prove that the sum of the first n cubic numbers is the square of the n-th triangular number by induction.

5) For $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n$ prove that $H_{2^n} \geq 1 + n/2$

6) Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

7) Prove the rational numbers are countable.

~~8)~~ Prove the real numbers are uncountable.

? 9) Show that a subset of a countable set is countable.

10) Give an epsilon-delta proof for the limit of a linear function. $ax+b$

11) Give an epsilon-delta proof for the limit of a quadratic funtion. ax^2+bx+c

$$\lim_{x \to d} ax^2+bx+c = a(d)^2+b(d)+c$$