Math 511

$\boxed{2.2.2}$ A $n \times n$ is singular iff $\det(A) = 0$

Notation

$\det(A) = |A|$ (Inductively Defined)

Basis: A is 1×1 $A = [a_{11}]$ $A^{-1} = \left[\frac{1}{a_{11}}\right]$

by observation $\det(A) = a_{11}$

and if $\det(A) = 0$ A^{-1} does not exist

Inductive Steps (finding new objects from older objects)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & - & \cdots & a_{nn} \end{bmatrix}$$ is $n \times n$ with $n = 2, 3, 4, 5, \ldots$

Definitions:

① Minor M_{ij} is the $(n-1) \times (n-1)$ matrix is A without row i and col. j.

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & 9 \end{bmatrix}$ $M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

② Cofactor $A_{ij} = (-1)^{i+j} \det(M_{ij})$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & & & & \\ \vdots & & & & \end{bmatrix}$$

ex $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$A_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

$= 45 - 48 = -3$

Note: if you collect all the cofactors into a Matrix

called the adjoint of A $= \begin{bmatrix} |M_{11}| & -|M_{12}| & |M_{13}| & \cdots \\ -|M_{21}| & |M_{22}| & \cdots & \\ & & & \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ & \vdots & \end{bmatrix}$

Induction Rule for Determinants.

| Cofactor Expansion to find the determinant |

$|A| =$ expand along one row or one col. and use a linear combination of the cofactors.

ex $\underset{3\times3}{A} = \begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}$

ex column 2 cofactor expansion.

$$\underset{3\times3}{|A|} = a_{12}|A_{12}| + a_{22}|A_{22}| + a_{32}|A_{32}|$$

has a det of 2x2

ex expand on row i

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \cdots + a_{in}A_{in}$$

ex expand on col. j

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \cdots + a_{nj}A_{nj}$$

$$A_{k\ell} = (-1)^{k+\ell}|M_{k\ell}|$$

ex
$$\begin{vmatrix} -1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 3 & 1 & -1 & 4 \\ 1 & 0 & 2 & 1 \end{vmatrix} = -0 \cdot \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 4 \\ 0 & 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 3 & 1 \\ 3 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} -1 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -1 & 2 & 3 \\ 3 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix}$$

$= \underline{\text{etc}}$

Expensive! $n \times n$ require $O(n!)$ multiplications

Hope! T is a triangular matrix.

ex (upper tria)
$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{bmatrix} = t_{11}\begin{vmatrix} t_{22} & t_{23} \\ 0 & t_{33} \end{vmatrix} - 0 + 0$$

$$= t_{11}t_{22}t_{33}$$

so $\boxed{|T| = t_{11} t_{22} \cdots t_{nn}}$

Hope of Hopes! -

$$\boxed{E_n \cdots E_2 E_1 A} = U$$

upper triangular.

$$|E^{T_3}| = \left| \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & \mu & 1 \end{bmatrix} \right| = 1$$

$$|E^{T_2}| = \left| \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \alpha & \\ & & & \ddots \\ 0 & & & & 1 \end{bmatrix} \right| = \alpha$$

$$|E^{T_1}| = \left| \left[I \text{ with rows swapped} \right] \right| = -1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$\boxed{\text{Type 2}}$ $\det(E^{T_2} A) = \alpha a_{i1} A_{i1} + \alpha a_{i2} A_{i2} + \dots$

$$\alpha \vec{a}_i \to \begin{bmatrix} \boxed{\alpha a_{i1} \quad \alpha a_{i2} \quad \cdots \quad \alpha a_{in}} \end{bmatrix} \qquad = \underbrace{\alpha}_{\det(E^{T_2})} \underbrace{(a_{i1} A_{i1} + \cdots}_{\det(A)}$$

$$¡ \det(E^{T_2} A) = \det(E^{T_2}) \det(A) !$$

$\boxed{\text{Type 1}}$ $\det(E^{T_1} A) = \underbrace{-}_{\det(E^{T_1}) = -1} \det(A)$

$$¡ \det(E^{T_1} A) = \det(E^{T_1}) \det(A) !$$

$\boxed{\text{Type 3}}$ $\det(E^{T_3} A) = \begin{vmatrix} \text{is } A \text{ but} \\ \text{New } r_i = r_i + M r_j \end{vmatrix}$

expand on this new row...

$$= (a_{i1} + M a_{j1}) A_{i1} + (a_{i2} + M a_{j2}) A_{i2} + \cdots$$

$\boxed{\text{Lemma}}$ any row in linear combination w/ some other row's cofactors is zero.

$$= \underbrace{1}_{\det(E^{T_3})} \cdot \underbrace{\det(A)}_{\det(A)}$$

So $\det(E_n \cdots E_2 E_1 A) = \det(U)$

$\det(E_n)\det(E_{n-1})\cdots\det(E_2)\det(E_1)\underline{\det(A)} = \det(U)$

$$\boxed{\det(A) = \frac{\det(U)}{\det(E_n)\cdots\det(E_2)\det(E_1)}}$$